1- In S_7 consider the two permutation $\sigma = (1 \quad 3 \quad 7)(2 \quad 6)$ and

$\tau = (2 \quad 4 \quad 7 \quad 6 \quad 5)$:

(i)Find $O(\sigma)$ and $O(\tau)$

(ii) Find $\sigma\tau$

(iii) If $\alpha = (1 \quad 4)(2 \quad 5)(6 \quad 7)$, find $\alpha\sigma\alpha^{-1}$

Solution: (i) $O(\sigma) = lcm(3,2) = 6$, $O(\tau) = 5$

(ii) $\sigma\tau = (2 \quad 4 \quad 1 \quad 3 \quad 7)(5 \quad 6)$

(iii) $\alpha\sigma\alpha^{-1} = (3 \quad 6 \quad 4)(5 \quad 7)$.

2- Stare Cayley's theorem

Solution: Any group is isomorphic to a permutation group.

3- Show that $\left|A_n\right| = \dfrac{\left|S_n\right|}{2}$.

4- State only on reason to assert that D_8 is not isomorphic to Q_8

Solution: Number of elements in D_8 of order 4 is equal to 2 while the Number of elements in Q_8 of order 4 is equal to 6.

5- Let $H \leq G$. When H is called normal subgroup of G.

Solution: H is normal subgroup of G if $aH = Ha$, for all $a \in G$.

6- Define a mapping $f:(\mathbb{Z},+) \to (\mathbb{Z},+)$ by $f(x) = 3x$

(i) Show that f is a homomorphism.

(ii) Find $Kerf$.

Solution: (i) $f(x+y) = 3(x+y) = 3x+3y = f(x)+f(y)$, for all $x, y \in \mathbb{Z}$.

(ii) $Kerf = \{x \in \mathbb{Z} : f(x) = 0\} = \{x \in \mathbb{Z} : 3x = 0\} = \{0\}$.

7- Write the multiplication table of the group $\{1, a, b, ab\}$ where $a^2 = b^2 = (ab)^2 = 1$.

Solution:

	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

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8-Find $\langle 2 \rangle, O(2)$ in the following groups:

1- $(\mathbb{Z}_{10}, +)$
2- $(\mathbb{Z}_{11}^*, \cdot)$
3- $(\mathbb{Z}, +)$
4- $(\mathbb{R}^*, \cdot)$

Solution:

1- $\langle 2 \rangle = \{0, 2, 4, 6, 8\}, \quad O(2) = 5$

2- $\langle 2 \rangle = \{1, 2, ..., 10\} = \mathbb{Z}_{11}^*, \quad O(2) = 10$ and in this case we say that $\mathbb{Z}_{11}^*$ is cyclic group generated by the element 2. It may there is more than one generator. For example the element 6 is also a generator of $\mathbb{Z}_{11}^*$. Why?. Is there other generators for this group?.

3- $\langle 2 \rangle = \left\{2^m = \underbrace{2 + 2 + ... + 2}_{m-times} : \text{m} \in \mathbb{Z}\right\} = \{..., -4, -2, 0, 2, 4, ...\} = 2\mathbb{Z}, \quad O(2) = \infty$.

4- $\langle 2 \rangle = \left\{2^m = \underbrace{2 \times 2 \times ... \times 2}_{m-times} : \text{m} \in \mathbb{Z}\right\} = \left\{..., \frac{1}{2^2}, \frac{1}{2}, 1, 2, 2^2, ...\right\}, \quad O(2) = \infty$.

9- Let $G=\left\{A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathrm{M}_2(\mathbb{R}), |A|\neq 0\right\}$ be the group of invertible matrices of $M_2(\mathbb{R})$ with respect to matrix multiplication

1- If $A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find $\langle A\rangle, \quad O(A)$

2- If $B=\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, find $\langle B\rangle, \quad O(\mathrm{B})$

3- If $C=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find $\langle C\rangle, \quad O(C)$.

Solution

1- $\langle A\rangle=\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, \quad O(A)=4$ **.**

2- $\langle B\rangle=\left\{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, \quad O(\mathrm{B})=3$

3- $\langle C\rangle=\left\{\ldots,\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ldots\right\}, \quad O(C)=\infty$ **.**

10-In S_6 , Find $\langle \alpha\rangle, \quad O(\alpha)$

1- $\alpha=(1\ \ 2\ \ 4)$

2- $\alpha=(2\ \ 5\ \ 1\ \ 6)$

3- $\alpha=(1\ \ 2\ \ 4\ \ 5\ \ 6)$

4- $\alpha=(1\ \ 4)(2\ \ 6)$

5- $\alpha=(1\ \ 3\ \ 5)(2\ \ 6)$

Solution

1- $\langle \alpha\rangle=\{e,(1\ \ 2\ \ 4),(1\ \ 4\ \ 2)\}, \quad O(\alpha)=3$

2- $\langle \alpha\rangle=\{e,(2\ \ 5\ \ 1\ \ 6),(1\ \ 2)(5\ \ 6),(2\ \ 6\ \ 1\ \ 5)\}, \quad O(\alpha)=4$

3- $\langle \alpha \rangle = \{e, (1\quad 2\quad 4\quad 5\quad 6), (1\quad 4\quad 6\quad 2\quad 5), (1\quad 5\quad 2\quad 6\quad 4), (1\quad 2\quad 4\quad 5\quad 6), (1\quad 6\quad 5\quad 4\quad 2)\}$

$O(\alpha) = 5$.

4- $\langle \alpha \rangle = \{\mathrm{e}, (1\quad 4)(2\quad 6)\}, \quad O(\alpha) = 2$.

5- $a = (1\quad 3\quad 5), \quad b = (2\quad 6)$

$\langle \alpha \rangle = \{\mathrm{e}, ab, (ab)^2 = a^2, (ab)^3 = b, (ab)^4 = a, (ab)^5 = a^2\,\mathrm{b}\}, (ab)^6 = e, O(\alpha) = 6$