Ministry of Education
Al-Imam Mohammed Ibn Saud Islamic University
College of Science
Department of Mathematics and Statistics

Final Examination

Course Name: Appl. Calculus 1
Course Code: MAT 113
Semester/Year: First/1437-1438 H
Date/Time: 17-04-1438 H / 8:00 am
Duration: 2 Hours

Instructions: Only ordinary calculators are allowed.

Model Answer

Question 1. [10= (2+3+2)+3 marks]: 1-Evaluate each of the following limits. All work must be shown.

$$(a)\quad \lim_{x\to 2^+}\frac{\left|x-2\right|}{x^2-x-2}\quad \left(\frac{0}{0}\right)\text{[0.5 mark]}$$

$$=\lim_{x\to 2^+}\frac{x-2}{(x-2)(x+1)}\text{[0.5 mark]}$$

$$=\lim_{x\to 2^+}\frac{1}{(x+1)}=\frac{1}{3}.\text{[1 mark]}$$

$$as\ we\ have\ \left|x-2\right|=\begin{cases}(x-2) & if\ x\ge 2\\ -(x-2) & if\ x<2\end{cases}$$

$$(b)\quad \lim_{x\to 0}\frac{\sin(3x)-3x-x^2}{1-\cos(2x)}\quad (\frac{0}{0})\text{[0.5 mark]}$$

Applying L'Hopital rule(= LR)

$$\lim_{x\to 0}\frac{\sin(3x)-3x-x^2}{1-\cos(2x)}=\lim_{x\to 0}\frac{3\cos(3x)-3-2x}{2\sin(2x)}\quad (\frac{0}{0})\text{[1 mark]}$$

$$\underset{LR}{=}\lim_{x\to 0}\frac{-9\sin(3x)-2}{4\cos(2x)}\text{[1 mark]}$$

$$=\frac{-9\sin(0)-2}{4\cos(0)}=\frac{-2}{4}=-\frac{1}{2}\text{[0.5 mark]}$$

(c) $\lim_{x\to\infty} \lim_{x\to\infty} \frac{x^2+x}{xe^x+x}$ $(\frac{\infty}{\infty})$ [0.5 mark], *Applying L'Hopital rule(= LR)*

$$\lim_{x\to\infty} \frac{x^2+x}{xe^x+x} \underset{LR}{=} \lim_{x\to\infty} \frac{2x+1}{xe^x+e^x+1} \quad (\frac{\infty}{\infty}) \quad \text{[1 mark]}$$

$$\underset{LR}{=} \lim_{x\to\infty} \frac{2}{xe^x+2e^x} = 0. \quad \text{[0.5 mark]}$$

2-Find the value of the constant c that makes the following function continuous

$$f(x) = \begin{cases} 2x + \frac{9}{x}, & if\ x \geq 3 \\ -4x + c, & if\ x < 3 \end{cases}$$

Solution

We need to study the continuity at the point where the function's definition is changing (Otherwise, $f(x)$ is continuous). That is at $x = 3$: [1 mark]

$f(3) = 9$,

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2x + \frac{9}{x} = 9$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} - 4x + c = -12 + c$. [1 mark]

In order to $f(x)$ is continuous at $x = 3$, it must $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$. That is $9 = -12 + c$ and $c = 21$. [1 mark]

Question 2. [**10= (3+2)+3+2 marks**]

1- Compute the first derivative for the following:

$$(a) \quad y = \ln(x^3 + 9) + \tan^{-1}(x^2).$$

Solution

$$y' = \frac{3x^2}{x^3+9} + \frac{2x}{1+x^4}. \qquad [1.5 \text{ mark}] + \qquad [1.5 \text{ mark}]$$

(b) $y = \sqrt{(x^2+1)(x^4+1)}$.

$$y' = \frac{2x(x^4+1) + 4x^3(x^2+1)}{2\sqrt{(x^2+1)(x^4+1)}}. \qquad [1 \text{ mark}] + \qquad [1 \text{ mark}]$$

2- Find $y'(x)$ for $x^2 - xy + y^2 = 7$. Then, find an equation of the tangent line at the point (-1,2).

Solution

First, we find $y'(x)$ implicitly. Differentiating both sides with respect to x, we get

$$2x - xy' - y + 2yy' = 0$$
$$-xy' + 2yy' = y - 2x$$
$$(-x+2y)y' = y - 2x$$
$$y' = \frac{y-2x}{-x+2y}. \qquad [1 \text{ mark}]$$

Second, we calculate the slope at the point (-1,2):

$$slope = y'\Big|_{(-1,2)} = \frac{2-2(-1)}{-(-1)+2(2)} = \frac{4}{5} [1 \text{ marks}]$$

Then. the equation of the tangent line at (-1,2) is

$$y = \frac{4}{5}(x+1) + 2 \qquad [1 \text{ mark}]$$

3- Find a value of c satisfying the conclusion of the Rolle's Theorem for $f(x) = x^3 - x + 1$ on [0, 1].

Solution

First, we verify that the hypotheses of the theorem are satisfied: f is differentiable and continuous for all x [since $f(x)$ is a polynomial and all polynomials are continuous and differentiable everywhere]. Also, $f(0) = f(1) = 1$. [1 mark]

We have

$$f'(x) = 3x^2 - 1$$

We now look for values of c such that

$$f'(c) = 3c^2 - 1 = 0$$

Solving this quadratic equation, we get $c = -\frac{1}{\sqrt{3}} \simeq -0.577$ [not in the interval (0, 1)]

and $c = \frac{1}{\sqrt{3}} \approx 0.577 \in (0,1)$. [1 mark]

Question 3.[**10 marks**] Consider the function $f(x) = x^4 - 6x^2 + 8x + 12$. Its derivative is $f'(x) = 4(x+2)(x-1)^2$.

(a) Find the critical numbers of $f(x)$. Determine the absolute extrema of $f(x)$ on the interval [0,2].

(b)Determine the intervals where the given function is increasing and decreasing.

(c) Determine all local extrema of the given function.

(d)Find the intervals where the graph of given function is concave up and concave down.

(e)Find the inflection points of $f(x)$.

Solution

Instructions: Only ordinary calculators are allowed.

(a) [$f'(x)=4(x+2)(x-1)^2=0$. Thus we have two critical numbers, $x=-2$ *and* $x=1$. [0.5 mark]]

Only one critical number is in the interval [0,2] which is $x=1$. So, we compare the values at the endpoints:

[$f(0)=12$, $f(2)=20$, and the value at the critical number: $f(1)=15$]. [0.5 mark]

Since f is continuous on [0,2], then the absolute extrema must be among these three values.

[Thus, $f(2)=20$ is the absolute maximum [0.5 mark]]and

[$f(0)=12$ is the absolute minimum. [0.5 mark]].

(b) We have: $f'(x)>0$ on (-2,1)[0.5 mark] ∪ (1,∞). Thus, f increasing[0.5 mark].

and [$f'(x)<0$ on (-∞ , -2). Thus, f decreasing. [1 mark]]

(c) From the above part (b), it follows from the First Derivative Test that f has a local minimum located at $x=-2$ *and* $f(-2)=-12$. [1 mark]

There no local maximum. [1 mark]

(d) We have $f''(x)=12x^2-12=12(x^2-1)$.

and we get,

$$f''(x)=12(x^2-1)\begin{cases} >0 & on\ (-\infty,-1)\cup(1,\infty) \quad Concave\ up \quad [1\ mark] \\ <0 & on\ (-1,1) \quad Concave\ down \quad [1\ mark] \end{cases}.$$

(e) From part (d), there are change of concavity at x=-1,1. Thus we have two inflection points:

(-1,f (-1))=(-1.-1) is an inflection point. [1 mark] Also (1,f (1))=(1,15) is an inflection point [1 mark].

Question 4. [10=8+2 marks]

1- Evaluate each of the following integrals, showing all reasoning.

$$(a) \quad \int_1^4 \frac{2x^2+x+4}{x}\,dx$$

Solution

$$(a) \quad \int_1^4 \frac{2x^2+x+4}{x}\,dx = \int_1^4 (2x+1+\frac{4}{x})dx = [x^2+x+4\ln|x|]_1^4 \quad \text{[1 mark]}$$

$$= [(4^2+4+\ln 4)-(1^2+1+\ln 1)] = 18+\ln 4 \quad \text{[1 mark]}$$

$$(b) \quad \int \frac{x^3}{x^4+1}\,dx$$

Solution

$$(b) \quad \int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\ln|x^4+1|+c \quad \text{[2 marks]}.$$ (You may use the substitution $u = x^4+1$)

$$(c) \quad \int x^2\cos(x^3+1)dx \ (\textit{Use a substitution with } u = x^3+1)$$

Solution

Put $u = x^3+1$. Thus $du = 3x^2dx$ [0.5 mark] and

$$\int x^2\cos(x^3+1)dx = \int \cos(u)\frac{dx}{3} = \frac{1}{3}\sin(u)+c = \frac{1}{3}\sin(x^3+1)+c \quad \text{[1.5 mark]}$$

Instructions: Only ordinary calculators are allowed.

(d) $\int 2x\sqrt{x^2-9}\,dx.$

Solution

Put $u = x^2 - 9$ and hence $du = 2xdx$ [0.5 mark]

$$\int 2x\sqrt{x^2-9}\,dx = \int \sqrt{u}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2(x^2-9)^{\frac{3}{2}}}{3} + c$$ [1.5 mark]

2- If $f(x) = \int_5^{x^2} \sqrt{1+t^2}\,dt$, find $f'(x)$.

Solution

$$f'(x) = \frac{d}{dx}\int_5^{x^2}\sqrt{1+t^2}\,dt = \frac{d}{dx^2}\int_5^{x^2}\sqrt{1+t^2}\,dt\,\frac{dx^2}{dx} = (1+(x^2)^2).2x = 2x(1+x^4)$$ [2 marks]

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