**Ministry of Education**
**Al-Imam Mohammed Ibn Saud Islamic University**
**College of Science**
**Department of Mathematics and Statistics**

# Model Answer

**Course Name:** Calculus I
**Course Code:** MAT 101
**Semester/Year:** Second/1436-1437
**Date:** 01-08-1437
**Duration:** 120 min's

**Instructions:** Only ordinary calculators are allowed.

**Question 1 [8=4x2 marks ]**: Evaluate each of the following limits. All work must be shown.

Solution

$$(a)\quad \lim_{x\to 2^+}\frac{|x-2|}{x^2-7x+10}=\lim_{x\to 2^+}\frac{x-2}{(x-2)(x-5)}=\lim_{x\to 2^+}\frac{1}{(x-5)}=-\frac{1}{3}.\,[1\text{ mark}]+[1\text{ mark}]$$

$$as\ |x-2|=\begin{cases}(x-2) & if\ x\geq 2\\ -(x-2) & if\ x<2\end{cases}$$

$$(b)\quad \lim_{x\to\infty}\frac{x^3+8x}{x^2+e^x}\quad (\frac{\infty}{\infty}),\quad Applying\ L'Hopital\ rule(=LR)$$

$$\lim_{x\to\infty}\frac{x^3+8x}{x^2+e^x}\underset{LR}{=}\lim_{x\to\infty}\frac{3x^2+8}{2x+e^x}\underset{LR}{=}\lim_{x\to\infty}\frac{6x}{2+e^x}\underset{LR}{=}\lim_{x\to\infty}\frac{6}{e^x}=0.\,[1\text{ mark}]+[1\text{ mark}]$$

$$(c)\quad \lim_{x\to 0}\frac{x^2}{e^x-x-1}\quad (\frac{0}{0}),$$

$$\lim_{x\to 0}\frac{x^2}{e^x-x-1}\underset{LR}{=}\lim_{x\to 0}\frac{2x}{e^x-1}\underset{LR}{=}\lim_{x\to 0}\frac{2}{e^x}=2.\,[1\text{ mark}]+[1\text{ mark}]$$

$$(d)\quad \lim_{x\to 0}\frac{1-\cos(4x)}{x^2}\quad (\frac{0}{0})$$

$$\lim_{x\to 0}\frac{1-\cos(4x)}{x^2}\underset{LR}{=}\lim_{x\to 0}\frac{4\sin(4x)}{2x}\underset{LR}{=}\lim_{x\to 0}\frac{16\cos(4x)}{2}=8.\,[1\text{ mark}]+[1\text{ mark}]$$

Another solution:

Multiplying the numerator and denominator **by** $1+\cos(4x)$**,** we get**,**

$$\lim_{x\to 0}\frac{1-\cos(4x)}{x^2}=\lim_{x\to 0}\frac{(1-\cos(4x))(1+\cos(4x))}{x^2(1+\cos(4x))}=$$

$$\lim_{x\to 0}\frac{\sin^2(4x)}{x^2(1+\cos(4x))}=\lim_{x\to 0}\left(\frac{\sin(4x)}{x}\right)^2.\frac{1}{(1+\cos(4x))}=4^2\times\frac{1}{2}=8.\left[2\text{ marks}\right]$$

**Question 2 [ 14=4x2+2+4 marks ]:**

(1) Find the first derivatives of the following functions:

$(a)\quad f(x)=x^{-3}+3^x$

$f'(x)=-3x^{-4}+3^x\ln 3.\left[1\text{ mark}\right]+\left[1\text{ mark}\right]$

$(b)\quad f(x)=\sin^{-1}(x^2)+\tan(x).$

$$f'(x)=\frac{2x}{\sqrt{1-x^4}}+\sec^2(x).\left[1\text{ mark}\right]+\left[1\text{ mark}\right]$$

$$(c)\quad f(x)=\frac{x^2\cos(x)}{(x^2+9)}$$

$$f'(x)=\frac{(x^2+9)[x^2(-\sin(x))+2x\cos(x)]-x^2\cos(x).2x}{(x^2+9)^2}\left[2\text{ marks}\right]$$

$(d)\quad f(x)=\ln(x^2+4)+8e^{-x^2}.$

$$f'(x)=\frac{2x}{x^2+4}-16xe^{-x^2}.\left[1\text{ mark}\right]+\left[1\text{ mark}\right]$$

(2) Find $y'$, if $y=(\sin x)^x$.

Solution

Taking the natural logarithm of both sides of the given equation, we get

$$\ln y = x\ln(\sin x)$$

Then, differentiate both sides of this last

equation with respect to x. Using the chain rule on the left side and the product rule on the right side, we get

$$\frac{y'}{y} = x\frac{\cos x}{\sin x} + \ln(\sin x)\left[1 \text{ mark}\right]$$

Solving for $y'$, we get

$$y' = y[x\frac{\cos x}{\sin x} + \ln(\sin x)] = (\sin x)^x[x\cot x + \ln(\sin x)]\left[1 \text{ mark}\right]$$

(3) Find the slope of the tangent line to the curve below at the point $(1,2)$

$$x^4y^2 + 6x^5 - y^3 + 2x = 4.$$

Solution

First, we find $y'(x)$ implicitly. Differentiating both sides with respect to $x$, we get

$$2x^4yy' + 4x^3y^2 + 30x^4 - 3y^2y' + 2 = 0$$
$$(2x^4y - 3y^2)y' = -(4x^3y^2 + 30x^4 + 2)$$
$$y' = \frac{-(4x^3y^2 + 30x^4 + 2)}{(2x^4y - 3y^2)}\left[2 \text{ marks}\right]$$

then we can find the slope at the point $(1,2)$:

$$slope = y'\Big|_{(1,2)} = \frac{-(4\times 2^2 + 30 + 2)}{(2\times 2 - 3\times 2^2)} = \frac{-48}{-8} = 6\left[2 \text{ marks}\right]$$

**Question 3 [ 10=2+4x2 marks]:**

(1) Find the absolute maximum and absolute minimum of $f(x) = \frac{x}{x^2+1}$ on the interval $[0,3]$.

Solution

First, we find the critical numbers in the given interval:

$f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$. So, the critical numbers is all x such that $1-x^2=0$. That is we have two critical numbers one only lie in the given interval which is at $x=1$. [1 mark] Now, we compute the function values at the critical number in the interval and at the endpoints of the interval:

$$f(0) = 0,$$
$$f(1) = \frac{1}{2},$$
$$f(3) = \frac{3}{10}.$$

Thus, $f(0)=0$, is the absolute minimum and $f(1)=\dfrac{1}{2}$, is the absolute maximum. [1 mark]

(2) Given the function : $f(x) = x^3 - 6x^2 + 9x + 1$

(a) Find all critical numbers.

(b) Find the intervals where the function is increasing or decreasing.

(c) Find the local extrema.

(d) Find the intervals where the graph of given function is concave up or concave down.

Solution

(a)

$f(x) = x^3 - 6x^2 + 9x + 1$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$\quad = 3(x-1)(x-3).$

Thus we have two critical numbers $x = 1, \quad x = 3.$ [2 marks]

(b) ( We may draw the real line )

We find that the given function: Increasing on $(-\infty, 1) \cup (3, \infty)$ and decreasing on $(1, 3)$. [2 marks]

(c) $f(1) = 1 - 6 + 9 + 1 = 5$ is a local maximum of the function

and $f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 + 1 = 1$ is a local minimum of the given function. [2 marks]

(d) $f''(x) = 6x - 12 = 6(x-2)$. ( We may draw the real line ). Thus the graph of given function is concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$. [2 marks]

**Question 4 [8=4x2 marks** ]:

(1) Evaluate each of the following integrals, showing all reasoning.

Solution

$$(a)\quad \int_2^3 \frac{2x^2+3x+1}{x}dx = \int_2^3 (2x+3+\frac{1}{x})dx = [x^2+3x+\ln|x|]_2^3 =$$
$$= [(9+9+\ln 3)-(4+6+\ln 2)] = 8+\ln 3-\ln 2.\text{[2 marks]}$$

$$(b)\quad \int \frac{x^3}{x^4+1}dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}dx = \frac{1}{4}\ln(x^4+1)+c.$$
$(\textit{Use the substitution } u = x^4+1)$ [2 marks]

$$(c)\quad \int 3x^2\sin(x^3+1)dx = -\cos(x^3+1)+c.\text{[2 marks]}$$

$$(d)\quad \int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int 2x\sqrt{x^2+1}\,dx = \frac{\frac{1}{2}(x^2+1)^{\frac{3}{2}}}{3/2}+c$$
$(\textit{Use the substitution } u = x^2+1)$ [2 marks]

**Extra question (Bonus)[ 4 marks]**: Given the function $f(x) = \dfrac{(x-1)^2}{(x+2)(x-4)}$ and its first derivative $f'(x) = \dfrac{18(1-x)}{(x+2)^2(x-4)^2}$ :

(a) Find all horizontal and vertical asymptotes, if any.

(b) Determine on what intervals $f(x)$ increasing or decreasing.

Solution

(a) Since as $x \to \pm\infty,\ \dfrac{1}{x} \to 0\ \textit{and}\ \dfrac{1}{x^2} \to 0,$ we get that $\lim\limits_{x\to\infty} \dfrac{(x-1)^2}{(x+2)(x-4)} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{(x-1)^2}{(x+2)(x-4)} = 1$. Thus, the line $y = 1$ is a horizontal asymptote. [1 mark]

The given function is not continuous at $x=-2,4$.

$$\lim_{x\to -2^+}\frac{(x-1)^2}{(x+2)(x-4)}=-\infty$$

$$\lim_{x\to -2^-}\frac{(x-1)^2}{(x+2)(x-4)}=\infty.$$

So, there is indeed a vertical asymptote at $x=-2$. $\left[\frac{1}{2}\ \text{mark}\right]$

Similarly for x=4

$$\lim_{x\to 4^+}\frac{(x-1)^2}{(x+2)(x-4)}=\infty$$

$$\lim_{x\to 4^-}\frac{(x-1)^2}{(x+2)(x-4)}=-\infty.$$

Thus, there exists a vertical asymptote a $x=4$. $\left[\frac{1}{2}\ \text{mark}\right]$

(b) The given function has only one critical number at $x=1$ $[1\ \text{mark}]$

And the function is increasing on $(-\infty,1)$ and is decreasing on $(1,\infty)$. $[1\ \text{mark}]$