Kingdom of Saudi Arabia
Ministry of Education
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المملكة العربية السعودية
وزارة التعليم
جامعة الإمام محمد بن سعود الإسلامية
-كلية العلوم-

Semester: 143820

MAT 106 **MIDTERM 2** **CALCULUS II**

Duration: 1H 15Min Section:__________ Student Name ____________________

Answers written outside the allocated space will NOT be graded. **Calculators are not allowed.**

Question:	1	2	3	Total
Points:	5	6	9	20
Score:				

1. $\boxed{\text{5 points}}$

(a) Evaluate the integral: $\displaystyle\int \frac{2x^3 - 7x - 6}{x^2 - 4}\,dx.$

$$\int \frac{2x^3-7x-6}{x^2-4}\,dx = \int 2x\,dx + \int \frac{x-6}{x^2-4}\,dx$$

$$\begin{array}{r} 2x \\ x^2-4\,\overline{)\,2x^3-7x-6} \\ \mp 2x^3 \pm 8x \\ \hline x-6 \end{array}$$

$$= x^2 + \int \frac{x-6}{x^2-4}\,dx$$

$$\frac{x-6}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$\Rightarrow x-6 = A(x+2) + B(x-2)$$

$$x=2 \Rightarrow -4 = 4A \Rightarrow \boxed{A=-1}$$

$$x=-2 \Rightarrow -8 = -4B \Rightarrow \boxed{B=2}$$

$$\int \frac{2x^3-7x-6}{x^2-4}\,dx = x^2 + \int \frac{-1}{x-2}\,dx + \int \frac{2}{x+2}\,dx$$

$$= \boxed{x^2 - \ln|x-2| + 2\ln|x+2| + C}$$

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(b) Determine wether the following integral converges or diverges: $\int_1^\infty \frac{1}{x(lnx)^2}\,dx$.

$$\int_1^\infty \frac{1}{x(\ln x)^2}dx = \lim_{R_1\to 1}\int_{R_1}^{2} \frac{1}{x(\ln x)^2}dx + \lim_{R_2\to\infty}\int_{2}^{R_2}\frac{1}{x(\ln x)^2}dx$$

$$u=\ln x \Rightarrow du = \frac{1}{x}dx \Rightarrow dx = x\,du$$

$$= \lim_{R_1\to 1}\int_{R_1}^{2} \frac{1}{\cancel{x}\,u^2}\cancel{x}\,du + \lim_{R_2\to\infty}\int_{2}^{R_2}\frac{1}{\cancel{x}\,u^2}\cancel{x}\,du$$

$$= \lim_{R_1\to 1}\frac{-1}{u} + \lim_{R_2\to\infty}\frac{-1}{u} = \lim_{R_1\to 1}\left(\frac{-1}{\ln x}\right)\Bigg|_{R_1}^{2} + \lim_{R_2\to\infty}\left(\frac{-1}{\ln x}\right)\Bigg|_{2}^{R_2}$$

$$= \lim_{R_1\to 1}\left(\frac{-1}{\ln 2}+\frac{1}{\ln R_1}\right) + \lim_{R_2\to\infty}\left(\frac{-1}{\ln R_2}+\frac{1}{\ln 2}\right)$$

$$= \frac{-1}{\ln 2} + \infty + \frac{-1}{\infty} + \frac{1}{\ln 2} = \infty \Rightarrow \boxed{\text{diverges}}$$

2. [6 points] Determine whether the following series converges or diverges:

(a) $\sum_{k=2}^{\infty}\frac{(k!)^2\,2^k}{(2k)!}$ by ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{((k+1)!)^2\,2^{k+1}}{(2k+2)!}\cdot\frac{(2k)!}{(k!)^2\,2^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{((k+1)\cancel{k!})^2\,\cancel{2^k}\,2}{(2k+2)(2k+1)\cancel{(2k)!}}\cdot\frac{\cancel{(2k)!}}{\cancel{(k!)^2}\,\cancel{2^k}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{2(k^2+2k+1)}{4k^2+6k+2}\right| = \frac{2}{4} < 1$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{(k!)^2\,2^k}{(2k)!}\quad \boxed{\text{Converges}}$$

(b) $\sum_{k=1}^{\infty} \frac{2k^2+1}{\sqrt{k^5+k^2}}$ by limit comparison test

$a_k = \frac{2k^2+1}{\sqrt{k^5+k^2}} > 0$, $b_k = \frac{k^2}{k^{5/2}} = \frac{1}{k^{1/2}} > 0$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^2+1}{\sqrt{k^5+k^2}} \cdot \frac{k^{1/2}}{1} = \lim_{k\to\infty} \frac{2k^{5/2}+k^{1/2}}{\sqrt{k^5+k^2}} \to \frac{\infty}{\infty}$

$= \lim_{k\to\infty} \frac{\frac{2k^{5/2}}{k^{5/2}} + \frac{k^{1/2}}{k^{5/2}}}{\sqrt{\frac{k^5}{k^5}+\frac{k^2}{k^5}}} = \lim_{k\to\infty} \frac{2+1/k^2}{\sqrt{1+1/k^2}}$

$= \frac{2+1/\infty}{\sqrt{1+1/\infty}} = 2 > 0$

since $\sum b_k$ diverges (by P-series) $\Rightarrow \sum_{k=1}^{\infty} \frac{2k^2+1}{\sqrt{k^5+k^2}}$ $\boxed{\text{diverges}}$

(c) $\sum_{k=3}^{\infty} \frac{k\,(-1)^k}{k^2+1}$ by alternating series test

* $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k^2+1} \to \frac{\infty}{\infty} \Rightarrow \lim_{k\to\infty} \frac{1}{2k} = \frac{1}{\infty} = 0$

* $a_k = \frac{k}{k^2+1}$ $\frac{\text{Positive}}{\text{Positive}}$ = positive > 0 $\forall\, k \geq 3$

* $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2}$

$= \frac{-x^2+1}{(x^2+1)^2}$ $\frac{\text{negative}}{\text{Positive}}$ = negative < 0 $\forall\, x \geq 3$

$\Rightarrow$ f(x) decreasing

$\Rightarrow \sum_{k=3}^{\infty} \frac{k(-1)^k}{k^2+1}$ $\boxed{\text{Converges}}$

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3. **9 points**

(a) Show that the following series converges, and find the sum of the series:

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k}-\frac{2}{5^{k-1}}\right)$$

$\sum_{k=1}^{\infty} \frac{3}{4^k}$ geometric series $a_1 = 3/4$ $r_1 = 1/4$

since $|r_1| = 1/4 < 1 \Rightarrow \sum_{k=1}^{\infty} 3/4^k$ converges to

$$\frac{a_1}{1-r_1} = \frac{3/4}{1-1/4} = \frac{3/4}{3/4} = \boxed{1}$$

$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}}$ geometric series $b_1 = 2$ $r_2 = 1/5$

since $|r_2| = 1/5 < 1 \Rightarrow \sum_{k=1}^{\infty} 2/5^{k-1}$ converges to

$$\frac{b_1}{1-r_2} = \frac{2}{1-1/5} = \frac{2}{4/5} = \boxed{10/4}$$

$$\Rightarrow \sum_{k=1}^{\infty}\left(3/4^k - 2/5^{k-1}\right) = 1 - 10/4 = \boxed{-6/4}$$

(b) Using the integral test to determine whether the series converges or diverges:

$$\sum_{k=1}^{\infty} k\, e^{-k^2}$$

* $f(x) = x e^{-x^2}$ contin. * contin $\Rightarrow$ f is continuous on $(1,\infty)$

* $f(x) = x e^{-x^2}$ positive * positive $\Rightarrow$ f is positive on $(1,\infty)$

* $f(x) = (1)e^{-x^2} + x(-2x e^{-x^2}) = e^{-x^2}(1-2x^2) < 0$ on $x \in (1,\infty)$

$\Rightarrow f(x)$ decreasing

$\lim_{R\to\infty}\int_1^R x e^{-x^2}\,dx$ let $u = -x^2 \Rightarrow du = -2x\,dx$

$\Rightarrow dx = \frac{du}{-2x}$

$$= \lim_{R\to\infty}\int \cancel{x} e^{u} \frac{du}{-2\cancel{x}} = \lim_{R\to\infty}\left(\frac{-e^{u}}{2}\right)$$

$$= \lim_{R\to\infty}\left(\frac{-e^{-x^2}}{2}\Big|_1^R\right) = \lim_{R\to\infty}\left(-\frac{e^{-R^2}}{2} + \frac{e^{-1}}{2}\right) = \frac{1}{\infty} + \frac{1}{2e} = \boxed{\frac{1}{2e}}$$

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$\Rightarrow$ Converges

$$\Rightarrow \sum_{k=1}^{\infty} k e^{-k^2} \quad \boxed{\text{Converges}}$$

(c) Determine the interval and radius of convergence of the following power series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k (x+2)^k}{k\ 2^k}$$

$$\lim_{k\to\infty} \left| \frac{(x+2)^{k+1}}{(k+1)\ 2^{k+1}} \cdot \frac{k\ 2^k}{(x+2)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(x+2)^k}\ (x+2)}{(k+1)\ \cancel{2^k}\ 2} \cdot \frac{k\ \cancel{2^k}}{\cancel{(x+2)^k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(x+2)k}{2k+2} \right| = \left| \frac{x+2}{2} \right| < 1$$

$$\Rightarrow \quad |x+2| < 2 \quad \Rightarrow \quad -2 < x+2 < 2$$

$$\Rightarrow -4 < x < 0$$

$\underline{\underline{x=0}}$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k\ \cancel{2^k}}{k\ \cancel{2^k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ by alternating series test

$\Rightarrow$ convergent

$\underline{\underline{x=-4}}$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k\ (-2)^k}{k\ 2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k\ (-1)^k\ \cancel{(2)^k}}{k\ \cancel{2^k}} = \sum_{k=1}^{\infty} \frac{1}{k}$

harmonic series

$\Rightarrow$ divergent

$\Rightarrow$ interval convergence $-4 < x \leq 0$

$\Rightarrow$ radius convergence $r = \left| \frac{0-(-4)}{2} \right| = 2$

Good Luck

THE END