1

Model Answer for Mid Term MAT 651

2nd Semester 1438

Answer Q 1

(a) There are three I's and 11 other letters. First we arrange the 11 other letters with a space between each one, in the begining and in the end. There

$$\frac{11!}{(2!)^4 (1!)^3}$$ [1 mark]

Since There 2 E's, 2 F's, 2 T's, 2 N's, 1 D, 1 R and 1 O.

The previous step Creates 12 empty spaces in which three I's may be arranged, then there are $\binom{12}{3}$ ways. [1 mark] Hence, the required number is

$$\binom{12}{3} \cdot \frac{11!}{(2!)^4} =$$ [1 mark]

$$= 548,856,000.$$

$\boxed{2}$

(b) $\binom{12}{3}+\binom{12}{4}+\cdots+\binom{12}{12}=2^{12}-\left[\binom{12}{0}+\binom{12}{1}+\binom{12}{2}\right]$

[2 marks]

$= 4096-79 = 4017$

(c) We have $g(x)=\frac{1}{1-x}$ is the generating function of the sequence $a_n=1$, hence

$$x g'(x) = \frac{x}{(1-x)^2}$$ is the generating $f^{\underline{n}}$ [1 mark] for the sequence $a_n=n$, and

$$x\left(x g'(x)\right)' = \frac{x(1+x)}{(1-x)^3}$$ is the [1 mark] generating function for the sequence $a_n=n^2$.

Hence the generating function of the sequence $a_n=n^2-1$ is

$$\frac{x(1+x)}{(1-x)^3}-\frac{1}{1-x}$$ [1 mark]

$$=\frac{3x-1}{(1-x)^3}$$

[3]

Answer Q2

(a) We have 6 identical oranges and
4 distinct apples, so
5 distinct boxes

\# of ways

$$\binom{6+5-1}{6} \cdot 5^4 = \binom{10}{4} \cdot 5^4 \quad [1 \text{ mark}]$$

For the 2^{nd} requirment, we have three cases:

Case 1: We put 2 oranges each in three boxes and 2 boxes each contain 2 apples,

this can be done in

$$\binom{5}{3} \frac{4!}{(2!)^2} \text{ ways} \quad \tfrac{1}{2} \text{ mark}$$

Case 2: Two boxes each contain 2 oranges
" boxes " " 1 orang
" " " " 1 apple
1 box contains 2 apples

This can be done in

$$\binom{5}{2}\binom{3}{2} \frac{4!}{2!\,1!\,1!} \text{ ways}$$

[1/2 mark]

[4] Case 3

one box contains 2 oranges
4 boxes each contain 1 apple
4 ~ ~ ~ 1 orange

This can be done in

$$\binom{5}{1}\binom{4}{4}\frac{4!}{(1!)^4}$$ [1/2 mark]

So, the total number of ways equal

$$\binom{5}{3}\frac{4!}{(2!)^2}+\binom{5}{2}\binom{3}{2}\frac{4!}{2!}+\binom{5}{1}\binom{4}{4}4!$$

[1/2 mark]

(b) Let $A=\{a_1, a_2, \dots, a_m\}$,
$B=\{b_1, b_2, \dots, b_n\}$

If $m<n$, there is no onto functions. Now, suppose that $m \geq n$ and U be the set of all functions from A to B. Then

$$|U| = n^m.$$

Let $A_i = \{f \in U : b_i \in R(f)\}$ $i=1,2,\dots,n$

$R(f)$ is the range of f. [1 mark]

Now $A_1^c \cap A_2^c \cap \dots \cap A_n^c = \{f \in U : b_1, b_2, \dots, b_n \in R(f)\}$

[1 mark]

[5]

$$|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |U - (A_1 \cup A_2 \cup \cdots \cup A_n)|$$

$$= |U| - (\alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n)$$

where $\alpha_1 = \sum_{i=1}^{n} |A_i| = \binom{n}{1}(n-1)^m$

$$\alpha_2 = \sum_{1 \le i < j \le n} |A_i \cap A_j| = \binom{n}{2}(n-2)^m$$

[1 mark]

$$\alpha_j = \binom{n}{j}(n-j)^m, \quad 1 \le j \le n-1$$

$$\alpha_n = 0.$$

Hence, the required number is

[1 mark]

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m \cdots (-1)^{n-1}\binom{n}{n-1}1^m$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n}{k}(n-k)^m$$

[6] Answer Q 3

(a) Let a_n be the number of ternary strings that contain two consecutive 0's. We could start with either a 1 or a 2 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-1}$ ways), or we could start with 01 or 02 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-2}$ ways), or we could start with 00 and follow with any ternary string of length $n-2$ (this can be done in 3^{n-2} ways). Therefore the recurrence relation is

[1 mark] [1 mark] [1 mark]

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

(b) Let $h(x)$ be the exponential generating function of d_n. That is,

$$h(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n$$

For the given recurrence relation divide it by $n!$ and multiply it by x^n, we get [1 mark]

$$\frac{d_n}{n!} x^n = \frac{n\, d_{n-1}}{n!} x^n + \frac{2^n}{n!} x^n$$

[7]

Taking the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty} \frac{d_n x^n}{n!} = x \sum_{n=1}^{\infty} \frac{d_{n-1}}{(n-1)!} x^{n-1} + \sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$$

$$\Rightarrow \quad h(x) - \frac{d_0}{1!} = x\,h(x) + \sum_{n=1}^{\infty} \frac{(2x)^n}{n!}$$

$$h(x) - x\,h(x) = 1 + e^{2x} - 1 \quad \text{[1 mark]}$$

$$h(x) = \frac{e^{2x}}{1-x} = \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{2^k}{k!} \right) x^n$$

[1 mark]

That is

$$\sum_{n=0}^{\infty} \frac{d_n}{n!} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{2^k}{k!} \right) x^n$$

$$\frac{d_n}{n!} = \sum_{k=0}^{\infty} \frac{2^k}{k!}$$

$$\Rightarrow \quad d_n = n! \sum_{k=0}^{\infty} \frac{2^k}{k!} \quad \text{[1 mark]}$$

$$= n! \left[1 + \frac{2}{1!} + \frac{2^2}{2!} + \dots + \frac{2^n}{n!} \right].$$

[8] Answer Q4

(a) $\sum_{k=1}^{\infty} k(k+1) = \sum_{k=1}^{n}\left(2\binom{k}{2} + 2\binom{k}{1}\right)$

$= 2\sum_{k=1}^{n}\binom{k}{2} + 2\sum_{k=1}^{n}\binom{k}{1}$

$= 2\binom{n+1}{3} + 2\binom{n+1}{2}$ [1 mark]

$= 2\,\frac{(n+1)n(n-1)}{3!} + 2\,\frac{(n+1)n}{2!}$

$= \frac{n(n+1)[2n-2+6]}{6}$

[1 mark]

$= \frac{n(n+1)(2n+4)}{6} = \frac{n(n+1)(n+2)}{3}$.

(b) If $x_1 \geq 0$, # of solutions is

$\binom{15+3-1}{3-1} = \binom{17}{2} = 136$

[1 mark]

If $x_1 \geq 6$, # of ways

$\binom{15-6+3-1}{3-1} = \binom{11}{2} = 55$

So, for $x_1 \leq 5$, $x_2, x_3 \geq 0$, # of ways

$= 136 - 55 = 81$. [1 mark]

[9] (c) For $a_n^{(h)}$: The characteristic equation is

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$

We have two char. roots $r_1 = 2$, $r_2 = -1$

Then $a_n^{(h)} = c_1 2^n + c_2 (-1)^n$ [1 mark]

For $a_n^{(p)}$: as 2 is a char. root, with multiplicity 1, hence, we may assume a particular solution of the form

$$a_n^{(p)} = \alpha n 2^n. \quad \text{Hence}$$

$$\alpha n 2^n = \alpha (n-1) 2^{n-1} + 2\alpha (n-2) 2^{n-2} + 2^n$$

divide by 2^{n-2}, we get

$$4\alpha n = 2\alpha (n-1) + 2\alpha (n-2) + 4$$
$$4\alpha n - 2\alpha n - 2\alpha n + 2\alpha + 4\alpha = 4$$
$$6\alpha = 4 \Rightarrow \alpha = \frac{2}{3}$$

hence $a_n^{(p)} = \frac{2}{3} \cdot n 2^n$ [1 mark]

Now, the solution

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= c_1 2^n + c_2 (-1)^n + \frac{2}{3} n 2^n.$$ [1 mark]

[10]

as $a_0 = 4 \Rightarrow \quad c_1 + c_2 = 4$

as $a_1 = 12 \Rightarrow \quad 2c_1 - c_2 + \frac{4}{3} = 12$

adding $\quad 3c_1 = 16 - \frac{4}{3} = \frac{48}{3} - \frac{4}{3} = \frac{44}{3}$

$\Rightarrow c_1 = \frac{44}{9}$, and $c_2 = 4 - \frac{44}{9}$

$= \frac{36}{9} - \frac{44}{9} = \frac{-8}{9}$

Hence $\quad a_n = \frac{44}{9} \cdot 2^n - \frac{8}{9}(-1)^n + \frac{2}{3} n \cdot 2^n$

$= \left(\frac{44}{9} + \frac{2}{3} n\right) 2^n + \frac{8}{9}(-1)^{n+1}.$

[1 mark]

Answer
Extra question

We must get the number of nonnegative integer solutions of the equation $x_1 + 2x_2 + 5x_3 = 101$ [1 mark]

For $x_3 = 0$, to 20, we get 21 equations

$x_1 + 2x_2 = 101$ has $\left\lceil \frac{102}{2} \right\rceil$ solutions

$x_1 + 2x_2 = 96 \qquad \left\lceil \frac{97}{2} \right\rceil$ solutions

$x_1 + 2x_2 = 91$

[1 mark] $\quad \vdots \qquad \vdots$

$x_1 + 2x_2 = 6$

$x_1 + 2x_2 = 1 \qquad \left\lceil \frac{2}{2} \right\rceil$ solutions

So, the total number of solutions is

(11) $\lceil \frac{102}{2} \rceil + \lceil \frac{97}{2} \rceil + \lceil \frac{92}{2} \rceil + \cdots + \lceil \frac{7}{2} \rceil + \lceil \frac{2}{2} \rceil$

$= 51 + 49 + 46 + \cdots + 4 + 1$

$= 100 + 90 + 80 + \cdots + 20 + 10 + 1$

$= \frac{10}{2}(110) + 1 = \underline{551}.$ [1 mark]