Kingdom of Saudi Arabia
Ministry of Education
Al-Imam Mohammed Ibn Saud Islamic University
- - - College of Science - - -
Department of mathematics & statistics

المملكة العربية السعودية
وزارة التعليم
جامعة الإمام محمد بن سعود الإسلامية
- - - كلية العلوم - - -
قسم الرياضيات و الاحصاء

FINAL EXAMINATION

MAT 102 **CALCULUS 2** **Semester 2- 2017/2018 (1438/1439 Heg.)**

STUDENT NAME: ______________ STUDENT ID: ______________ DURATION: 2 HOURS

Question:	1	2	3	4	Total
Points:	12	6	14	8	40
Score:					

No calculators allowed. Answers written outside the allocated space will NOT be graded!!!

1. $\boxed{12 \text{ points}}$

1. Evaluate the following integrals

(a) $\int_0^1 x \tan^{-1} x \, dx$ **(3pts)**

Using integration by parts, let

$u = \tan^{-1} x$, $dv = x\,dx$

$\Rightarrow du = \frac{1}{1+x^2}\,dx$, $v = \frac{x^2}{2}$ $\boxed{1}$

Hence, $\int x \tan^{-1} x \, dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(\frac{1+x^2-1}{1+x^2}\right) dx$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$ $\boxed{1}$

Hence $\int_0^1 x \tan^{-1} x \, dx = \left[\left(\frac{x^2}{2}+\frac{1}{2}\right)\tan^{-1} x - \frac{1}{2}x\right]_0^1$ $\boxed{1}$

$= \left[\left(\tan^{-1} 1 + \frac{1}{2}\right) - \left(\left(0+\frac{1}{2}\right)\tan^{-1} 0 - \frac{1}{2}\cdot 0\right)\right] = \pi/4 - \frac{1}{2}$.

(b) $\displaystyle\int \frac{2}{x^3+2x^2+2x}\,dx$ **(2pts)**

Using the partial fractions,

$$\frac{2}{x^3+2x^2+2x} = \frac{2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

(since x^2+2x+2 has no real factorization.)

multiplying both sides by $x(x^2+2x+2)$, we get

$$2 = A(x^2+2x+2) + x(Bx+C)$$

We match up the coefficients of like powers of x:

$A+B=0$, $2A+C=0$, and $2A=2$. We get

$A=1$, $B=-1$, $C=-2$. Hence $\boxed{0.5}$

$$\int \frac{2}{x(x^2+2x+2)}\,dx = \int \left(\frac{1}{x} - \frac{(x+2)}{x^2+2x+2}\right)dx \quad \boxed{0.5}$$

$$= \ln|x| - \frac{1}{2}\int \left(\frac{2x+2+2}{x^2+2x+2}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2x+2| - \tan^{-1}(x+1) + C$$

(As $x^2+2x+2 = (x+1)^2+1$). $\boxed{1}$

(c) $\displaystyle\int_{-\pi/3}^{0} \sqrt{\cos x}\ \sin^3 x\ dx$ **(2pts)**

Let $u = \cos x \Rightarrow du = -\sin x\,dx$. $\boxed{0.5}$

As $x = -\pi/3 \Rightarrow u = 1/2$, and as $x=0 \Rightarrow u=1$.

$$\int_{-\pi/3}^{0} \sqrt{\cos x}\ \sin^3 x\,dx = \int_{-\pi/3}^{0} \sqrt{\cos x}\,(1-\cos^2 x)\sin x\,dx$$

$$= \int_{1/2}^{1} \sqrt{u}\,(1-u^2)(-du) = \int_{1/2}^{1} (u^{5/2} - u^{1/2})\,du$$

$$= \left[\frac{2u^{7/2}}{7} - \frac{2u^{3/2}}{3}\right]_{1/2}^{1} \quad \boxed{1}$$

$$= \left(\frac{2}{7} - \frac{2}{3}\right) - \left(\frac{2}{7}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{1/2} - \frac{2}{3}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{1/2}\right)$$

$$= -\frac{8}{21} - \left(\frac{\sqrt{2}}{56} - \frac{\sqrt{2}}{6}\right) \quad \boxed{0.5}$$

$$= -\frac{8}{21} + \frac{25}{168}\sqrt{2}.$$

(d) $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$ (2pts)

$$I = \int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}}dx + 2\int \frac{dx}{\sqrt{4-x^2}}$$

$$= -\sqrt{4-x^2} + 2\int \frac{dx}{\sqrt{4-x^2}} \quad \boxed{0.5}$$

put $x = 2\sin\theta$, for $-\frac{\pi}{2} < \theta < \pi/2$, we get $\boxed{0.5}$

$$I = -\sqrt{4-x^2} + 2\int \frac{2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}} = -\sqrt{4-x^2} - 2\int \frac{2\cos\theta}{2\cos\theta}d\theta$$

$$= -\sqrt{4-x^2} - 2\int d\theta = -\sqrt{4-x^2} - 2\theta$$

$$= -\sqrt{4-x^2} - 2\sin^{-1}\left(\frac{x}{2}\right) + C \quad \boxed{1}$$

2. Determine the convergence or divergence of $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}\,dx$ (3pts)

Substitue $u = \sqrt{x}$.

$$\int \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}dx = \int 2e^{-u}du = -2e^{-u} + C$$

$$= -\frac{2}{e^{\sqrt{x}}} + C \quad \boxed{1}$$

Hence $$\int_0^1 \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}dx = \lim_{R\to 0^+}\int_R^1 \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}dx + \lim_{R\to\infty}\int_1^R \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}dx$$

$$= \lim_{R\to 0^+}\left[\frac{-2}{e^{\sqrt{x}}}\right]_R^1 + \lim_{R\to\infty}\left[\frac{-2}{e^{\sqrt{x}}}\right]_1^R \quad \boxed{1}$$

$$= \lim_{R\to 0^+}\left[\frac{-2}{e} + \frac{2}{e^{\sqrt{R}}}\right] + \lim_{R\to\infty}\left[\frac{-2}{e^{\sqrt{R}}} + \frac{2}{e}\right]$$

$$= \left(\frac{-2}{e} + 2\right) + \left(0 + \frac{2}{e}\right) = 2 \quad \boxed{1}$$

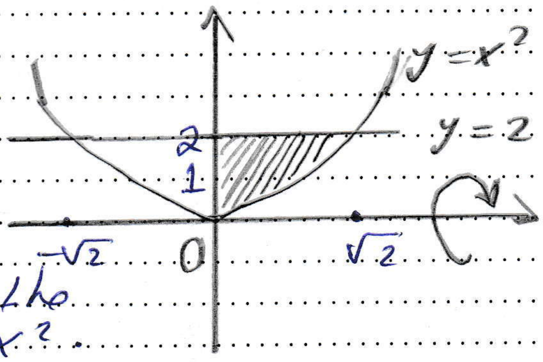
2. $\boxed{\text{6 points}}$

1. Let R be the region in the first quadrant bounded by the graphs of $y = x^2$, $x = 0$ and $y = 2$. Compute the volume of the solid formed by revolving R about

(a) the x-axis and (b) the line $y = 4$ (3pts)

The line $y=2$ intersect the curve $y=x^2$ at $x=\pm\sqrt{2}$.

(a) The outer radius r_o, is the distance from the x-axis to the line $y=2$. That is $r_o = 2$. The inner radius, r_i, is the distance from the x-axis to the curve $y=x^2$.



That is $r_i = x^2$. Hence, the required volume

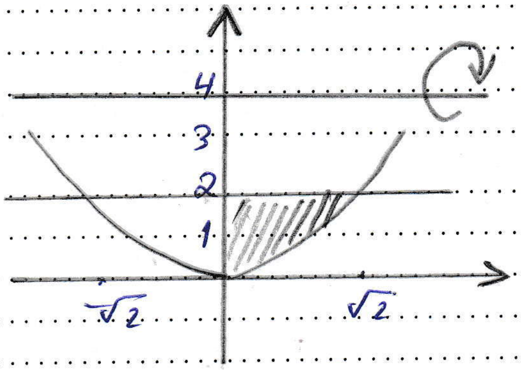
$$\pi\int_0^{\sqrt{2}}\left(2^2-(x^2)^2\right)dx = \pi\int_0^{\sqrt{2}}(4-x^4)\,dx \quad \boxed{1}$$

$$= \pi\left[4x-\frac{x^5}{5}\right]_0^{\sqrt{2}} = \pi\left(4\sqrt{2}-\frac{(\sqrt{2})^5}{5}\right)$$

$$= \pi\,\frac{16}{5} = \frac{16}{5}\pi \quad \boxed{1}$$

(b) $r_o = 4 - x^2$,

$r_i = 4-2 = 2$, the required volume,



$$\pi\int_0^{\sqrt{2}}\left[(4-x^2)^2-2^2\right]dx \quad \boxed{1}$$

$$= \pi\int_0^{\sqrt{2}}(16-8x^2+x^4-4)\,dx$$

$$= \pi\int_0^{\sqrt{2}}(12-8x^2+x^4)\,dx = \pi\left[12x-\frac{8x^3}{3}+\frac{x^5}{5}\right]_0^{\sqrt{2}}$$

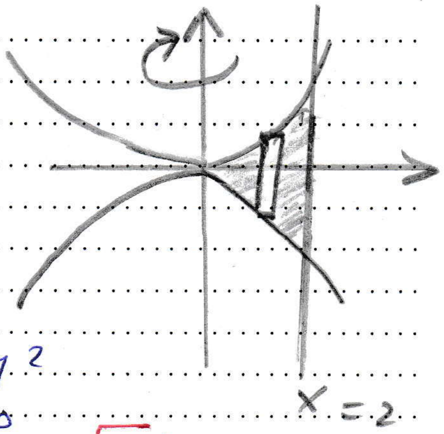
$$= \pi\left[12\sqrt{2}-\frac{8(\sqrt{2})^3}{3}+\frac{(\sqrt{2})^5}{5}\right] = \pi\left(\frac{180-80+12}{15}\right)\sqrt{2} \quad \boxed{1}$$

$$= \frac{112}{15}\pi\cdot\sqrt{2}$$

2. Use the method of cylindrical shells to find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2$, $y = -x^2$ and $x = 2$, about the y-axis. **(3pts)**

radius $= x$

height $= x^2 - (-x^2) = 2x^2$



$\boxed{1}$

$$V = \int_0^2 2\pi\, x\, (2x^2)\, dx$$

$$= 4\pi \int_0^2 x^3\, dx = 4\pi \left[\frac{x^4}{4}\right]_0^2$$

$\boxed{1}$

$$= 16\pi.$$

3. 14 points

1. Investigate the convergence or divergence of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos k}{e^k}$ **(2pts)**

To test for absolute convergence we consider the series of absolute values, $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\cos k}{e^k}\right|$.

As $\left|\frac{\cos k}{e^k}\right| = \frac{|\cos k|}{e^k} \le \frac{1}{e^k}$, $\boxed{1}$

since $|\cos k| \le 1$, for all k, and as $\displaystyle\sum_{k=1}^{\infty} \frac{1}{e^k}$ converges (by using the geometric series or by using the ratio test) $\Rightarrow$ By using the comparison test $\boxed{0.5}$ $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\cos k}{e^k}\right|$ converges too.

Consequently, the original series converges absolutely, and hence converges. $\boxed{0.5}$

(5)

(b) $\sum_{k=1}^{\infty} \left(\frac{k+10}{2k+1}\right)^{2k}$ (2pts)

Let $a_k = \left(\frac{k+10}{2k+1}\right)^{2k}$.

Using the root test, we get $\boxed{1}$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{k+10}{2k+1}\right|^{2k}} = \lim_{k\to\infty} \left(\frac{k+10}{2k+1}\right)^2$$

$= \frac{1}{4} < 1$. Hence the series is absolutely convergent. $\boxed{1}$

(c) $\sum_{k=1}^{\infty} \frac{k^{10}}{10^k}$ (2pts)

Let $a_k = \frac{k^{10}}{10^k}$. Using the ratio test, we get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^{10}\ 10^k}{10^{k+1}\ k^{10}}\right| \quad \boxed{1}$$

$$= \frac{1}{10} \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{10} = \frac{1}{10} < 1.$$

Hence the given series is absolutely convergent. $\boxed{1}$

2. Determine the radius and interval of convergence of $\sum_{k=2}^{\infty} \frac{x^k}{4^k\ \ln k}$ (4pts)

Let $a_k = \frac{x^k}{4^k \ln k}$. From the ratio test, we have

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{x^{k+1}\ 4^k \ln k}{4^{k+1} \ln(k+1)\ x^k}\right|$$

$$= \lim_{k\to\infty} \frac{1}{4}|x| \frac{\ln(k+1)}{\ln k}$$

$$= \frac{1}{4}|x| \cdot 1 \quad \boxed{1} \quad \left(\text{as } \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} = 1\right)$$

That is the series converges absolutely for $|x| < 4$ and diverges for $|x| > 4$. It remain only to test the endpoints of the interval: $x = \pm 4$. $\boxed{1}$

Using L'Hopital rule

$$\left(\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1\right)$$

For $x=4$, we have $\sum_{k=2}^{\infty} \frac{x^k}{4^k \ln k} = \sum_{k=2}^{\infty} \frac{1}{\ln k}$

as $k > \ln k$ for every $k \geq 2$, we get $\frac{1}{k} < \frac{1}{\ln k}$,

and as $\sum \frac{1}{k}$ diverges, then using the Comparison test, we get $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges too. $\boxed{0.5}$

For $x=-4$, we have $\sum_{k=2}^{\infty} \frac{x^k}{4^k \ln k} = \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ $\boxed{0.5}$

which is convergent using the alternating series test. Hence, the interval of convergent is $[-4, 4)$ and the radius of convergent is $r=4$. $\boxed{1}$

3. Find the Taylor series for $f(x) = \cos x$, expanded about $x = \pi/2$ and determine its radius and interval of convergence. **(4pts)**

$f(x) = \cos x$, $f(\pi/2) = 0$

$f'(x) = -\sin x$, $f'(\pi/2) = -1$

$f''(x) = -\cos x$, $f''(\pi/2) = 0$ $\boxed{1}$

$f'''(x) = \sin x$, $f'''(x) = 1$

$f^{(4)}(x) = \cos x$, $f^{(4)}(x) = 0$. Therefore $\cos x =$

$= -(x - \pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \frac{(x-\pi/2)^7}{7!} + \cdots$

$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x - \pi/2)^{2k+1}$ $\boxed{1}$. By the ratio test,

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} (x - \pi/2)^2 \frac{(2k+1)!}{(2k+3)!} = 0$ $\boxed{1}$

Hence the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is ∞. $\boxed{1}$

4. 8 points

1. Consider the parametric curve given by $x = 2 + 3\cos t$, $y = 3 + 2\sin t$

(a) Find an x-y equation for the curve. **(2pts)**

put $\frac{x-2}{3} = \cos t$, $\frac{y-3}{2} = \sin t$. 1

As $\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$

$\Rightarrow \frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$, 1 which means that the plane curve lies on the ellipse with center $(2,3)$.

(b) Identify all points at which the curve has a horizontal or vertical tangent line. **(2pts)**

The horizontal tangent lines occur where

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2\cos t}{-3\sin t} = 0$, which occur only when $2\cos t = 0$, provided that $x'(t) = -3\sin t \neq 0$. 1

Since $\cos t = 0$ only when $t = (2n+1)\pi/2$, $n \in \mathbb{Z}$,

The corresponding points on the curve are

$(x(\pi/2), y(\pi/2)) = (2,5)$; $(x(3\pi/2), y(3\pi/2)) = (2,1)$. 1

For vertical tangent lines, we need $\sin t = 0$, provided that $\cos t \neq 0$. So $t = n\pi$, $n \in \mathbb{Z}$ 1

(Clearly that $\cos(n\pi) \neq 0$). The corresponding points are

$(x(\pi), y(\pi)) = (-1,3)$,

$(x(2\pi), y(2\pi)) = (5,3)$. 1

2. Find all polar coordinate representation for the rectangular point $(2, 2\sqrt{3})$. **(2pts)**

with $x=2$ and $y=2\sqrt{3}$, we have $r^2 = x^2+y^2 = 2^2+(2\sqrt{3})^2$

$= 16 \Rightarrow r = \pm 4$. Also, $\tan\theta = y/x = \frac{2\sqrt{3}}{2} = \sqrt{3}$ 1

$\Rightarrow \theta = \pi/3$. All of the polar points must have the form

$(4, \pi/3 + 2n\pi)$ or 1 $(-4, \pi/3 + \pi + 2n\pi)$, for

any integer n

- -BONUS - -

Identify the center, foci and vertices of the ellipse $25x^2 + 4y^2 - 100x + 8y + 4 = 0$ **(3pts)**

$25x^2 - 100x + 4y^2 + 8y = -4 \Rightarrow$

$25(x^2-4x) + 4(y^2+2y) = -4 \Rightarrow$

$25(x^2-4x+4) + 4(y^2+2y+1) = -4+100+4$

$\Rightarrow 25(x-2)^2 + 4(y+1)^2 = 100$. Divide by 100

for both sides, we get $\frac{(x-2)^2}{4} + \frac{(y+1)^2}{25} = 1$ 1

Thus, we have $\frac{(x-2)^2}{2^2} + \frac{(y+1)^2}{5^2} = 1$,

The center is at $(2,-1)$. As $c^2 = a^2 - b^2 = 21$ 05

$\Rightarrow c = \sqrt{21}$. The major axis is parallel to the y-axis.

The foci are at $(2, -1+\sqrt{21})$ and $(2, -1-\sqrt{21})$ 1

The vertices are at $(2,4)$ and $(2,-6)$. 05

THE END