1

Key Answer Final Exam

MAT 651

2^{nd} Semester 216/2017.

Answer question 1:

(a) For each multiset of length 5 out of the set $\{1, 2, 3, \dots, 9\}$, there is EXACTLY one non-descending number. For example the multiset

$$\{1, 2, 2, 5, 7\}$$ [1 mark]

coresponds the non-descending number 12257, and conversely, any non-descending number coresponds a multiset. Thus, the total number of non-descending numbers is

$$\binom{9-1+5}{5} = \binom{13}{5} \quad \text{[2 marks]}$$

$$= \frac{13 \times \cancel{12} \times 11 \times \cancel{10} \times 9}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = 13 \times 99 = 1287$$

(b) There are 2 C's; 2 O's; 1 M's; 1 B's; 2 I's; 1 N's; 1 A's; 1 T's; 1 R's and 1 S's

Thus There $\frac{13!}{(2!)^3 (1!)^7} = 778,377,600$ [1 mark] = permutations of the letters of Combinatorics.

2 Fixing an O in both the begining and ending locations, we have 11 other letters to freely permute. This can be done in [1 mark]

$$\frac{11!}{(2!)^2(1!)^7} = \frac{11!}{4} = 9,979,200$$

- There are 8 consonants and 5 vowels. place the consonants like so:

... C ... C ... C ... C ... C ... C ... C ... C ...

The consonants can be placed in those 8 locations in $\frac{8!}{2!\,(1!)6}$. Now, consider placing the vowels in the underlined gaps (...). The two O's can be placed in $\binom{9}{2}$ locations, since we will choose 2 out of the 9 locations for the O's. The two I's can be placed in $\binom{7}{2}$ locations and finally the one A's can be placed in $\binom{5}{1}$ locations.

Using the multiplication principle, we get

$$\frac{8!}{2!} \times \binom{9}{2}\binom{7}{2}\binom{5}{1} \text{ ways.}$$ [1 mark]

$2!\,2!\,1!$

now think of them as one new letter to be arrangend with the remaining consonants in

$$\frac{9!}{2!\,(1!)^7} \text{ ways}$$

Thus, the total number of ways is

$$\frac{5!}{2!\,2!} \cdot \frac{9!}{2!} = 5{,}443{,}200.$$

(c) Every cycle in a bipartite graph is even and alternates between the vertices from the two partitions of $K_{m,n}$, V_1 and V_2. Since a Hamiltonian cycle uses all the vertices in V_1 and V_2, we must have $m = |V_1| = |V_2| = n$.

Suppose that $K_{n,n}$ has partite sets

[2 marks]

$$V_1 = \{u_1, u_2, \ldots, u_n\}$$
$$V_2 = \{v_1, v_2, \ldots, v_n\}.$$ Then

$u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_1$ is a Hamilton cycle.

Thus $K_{m,n}$ is Hamiltonian $\iff m=n$.

Answer question 2

Let $g(x)$ be the "gf" of a_n

(a) multiply the given recurrence relation by x^n, we get $a_n x^n = 2a_{n-1}x^n + 3^{n-1}x^n + nx^n$

Taking the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1}x^n + \sum_{n=1}^{\infty} 3^{n-1}x^n + \sum_{n=1}^{\infty} nx^n$$

$$g(x) - a_0 = 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 3^{n-1}x^{n-1} + \sum_{n=0}^{\infty} nx^n$$

$$\Rightarrow g(x) - a_0 = 2x\,g(x) + \frac{x}{1-3x} + \frac{x}{(1-x)^2}$$

$$(1-2x)g(x) = \frac{x}{1-3x} + \frac{x}{(1-x)^2}$$

$$= \frac{x[(1-x)^2 + (1-3x)]}{(1-3x)(1-x)^2}$$

$$= \frac{x^3 - 5x^2 + 2x}{(1-3x)(1-x)^2}$$

$$g(x) = \frac{x^3 - 5x^2 + 2x}{(1-2x)(1-3x)(1-x)^2}$$

[5] Using the partial fractions:

$$\frac{x^3-5x^2+2x}{(1-2x)(1-3x)(1-x)^2} = \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-x} + \frac{D}{(1-x)^2}$$

Then

$$A(1-3x)(1-x)^2 + B(1-2x)(1-x)^2 + C(1-x)(1-2x)(1-3x)$$
$$+ D(1-2x)(1-3x) = x^3-5x^2+2x$$

At $x=\frac{1}{2}$ $\Rightarrow$ $A\left(1-\frac{3}{2}\right)\left(1-\frac{1}{2}\right)^2 = \frac{1}{8} - \frac{5}{4} + 2\cdot\frac{1}{2}$,

$$A\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1-10+8}{8} = -\frac{1}{8}$$

$$\boxed{A=1}$$

At $x=\frac{1}{3}$ $\Rightarrow$ $B\left(1-\frac{2}{3}\right)\left(1-\frac{1}{3}\right)^2 = \frac{1}{27} - \frac{5}{9} + \frac{2}{3}$,

$$B\cdot\left(\frac{4}{27}\right) = \frac{1-15+18}{27} = \frac{4}{27}$$

$$\boxed{B=1}$$

At $x=1$: $D(1-2)(1-3) = 1-5+2$

$$2D = -2 \Rightarrow$$

$$\boxed{D=-1}$$

[6] <u>The constant term</u>: $(x=0)$

$$A+B+C+D=0$$

$$C=-A-B-D=-1-1+1=-1$$

<u>That is</u>

$$g(x)=\frac{1}{1-2x}+\frac{1}{1-3x}-\frac{1}{1-x}-\frac{1}{(1-x)^2}$$

$$=\sum_{n=0}^{\infty}2^nx^n+\sum_{n=0}^{\infty}3^nx^n-\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}\binom{n+1}{n}x^n$$

$$\sum a_nx^n=\sum\left(2^n+3^n-1-(n+1)\right)x^n$$

$$\Rightarrow a_n=2^n+3^n-1-n-1$$

$$=2^n+3^n-n-2.$$

(b) You may use the dihedral group D_8 of 16 elements (The group of symmetries of octagon). Or you may use the Neclace theorem

as

# of orbits (= number of colorings)

$$=\frac{1}{2n}\left(\sum_{d|n}\phi(d)\,k^{n/d}\right)+\frac{1}{4}\left(k^{\frac{n+2}{2}}+k^{\frac{n}{2}}\right)$$

[1 mark]

For n even

7 Here $n=8$

$$\# \text{ of colorings} = \frac{1}{16}[k^8 + k^4 + 2k^2 + 4k] + \frac{1}{4}[k^5 + k^4]$$ [2 marks]

If $k=2 \Rightarrow$

$$\# \text{ of colorings} = \frac{1}{16}[2^8 + 2^4 + 2^3 + 2^3] + \frac{1}{4}[2^5 + 2^4]$$

$$= \frac{2^4}{16}[2^4 + 2] + \frac{2^2}{4}[2^3 + 2^2]$$

$$= 18 + 12 = 30$$ [1 mark]

Answer question 3

(a) $\binom{n}{0} + 2\binom{n}{1} + \cdots + (n+1)\binom{n}{n} = \sum_{k=0}^{n} (k+1)\binom{n}{k}$

$$= \sum_{k=0}^{n} k\binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k}$$

$$= n2^{n-1} + 2^n.$$ [2 marks]

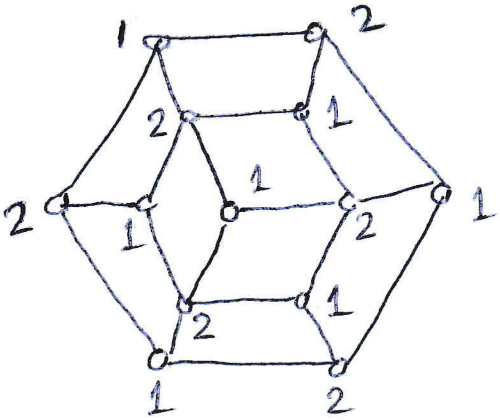
(We used that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, then differentiate both sides with respect to x,

$n(1+x)^{n-1} = \sum k\binom{n}{k}x^{k-1}, \ldots$)

$, \ldots, x=1.$

[8] (b)

(i)



$\Rightarrow \chi(G) = 2$

[1 mark]

Thus G is a bipartite.

(ii) G is not Hamiltonian. [1 mark]

Suppose G is Hamiltonian, then G has a cycle contain all its vertices, but number of vertices of G is 13. That is, we obtained an odd cycle which contradicts that G is a bipartite. [1 mark]

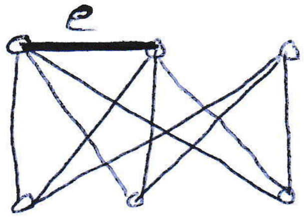
(iii) Let m and $\overline{m}$ be the number of edges in G and $\overline{G}$ respectively. We have

$$m = 21, \quad \overline{m} = \binom{13}{2} - 21 = 78 - 21 = 57$$

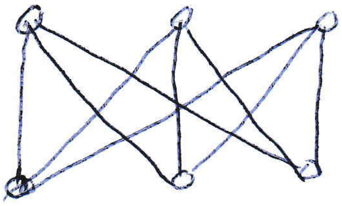
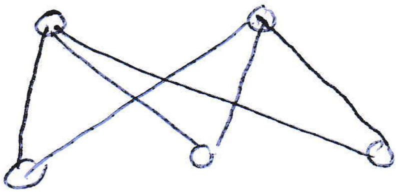
In $\overline{G}$ as $m = 57 > 3n - 6 = 3 \times 13 - 6 = 33$, then $\overline{G}$ is non-planar [1 mark]

9

(c) we may add the edge e in any of the two parts

$G =$ 

Using the theorem of deleting and contracting, we get

$=$  $-$ 

$$\Rightarrow P_G(k) = P_{K_{3,3}}(k) - P_{K_{2,3}}(k)$$ 2[marks]

$$= \left[k(k-1)^3 + 3k(k-1)(k-2)^3 + k(k-1)(k-2)(k-3)^3\right]$$

$$- \left[k(k-1)^3 + k(k-1)(k-2)^3\right]$$

$$= 2k(k-1)(k-2)^3 + k(k-1)(k-2)(k-3)^3.$$

[2 marks]

[10]

(a) Let A_{K_5} be the adjacency matrix of K_5

$$A_{K_5} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}_{5\times 5}$$

$$|\lambda I - A_{K_5}| = \begin{vmatrix} \lambda & -1 & -1 & -1 & -1 \\ -1 & \lambda & -1 & -1 & -1 \\ -1 & -1 & \lambda & -1 & -1 \\ -1 & -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & -1 & \lambda \end{vmatrix} \quad r_1 + (r_2 + r_3 + r_4 + r_5)$$

$$= \begin{vmatrix} \lambda-4 & \lambda-4 & \lambda-4 & \lambda-4 & \lambda-4 \\ -1 & \lambda & -1 & -1 & -1 \\ -1 & -1 & \lambda & -1 & -1 \\ -1 & -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & -1 & \lambda \end{vmatrix}$$

[1 mark]

$$= (\lambda - 4) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & \lambda & -1 & -1 & -1 \\ -1 & -1 & \lambda & -1 & -1 \\ -1 & -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & -1 & \lambda \end{vmatrix} \quad \begin{matrix} r_2 + r_1 \\ r_3 + r_1 \\ r_4 + r_1 \\ r_5 + r_1 \end{matrix}$$

[11]
$$(\lambda-4)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda+1 & 0 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda-4)(\lambda+1)^4.$$

$$\Rightarrow (\lambda-4)(\lambda+1)^4 = 0$$

$$\mathrm{Spect}(K_5) = \{4, (-1)^4\}.$$ (1 mark)

(b) The two given graphs are not isomorphic, since, the two vertices of degree 3 in G are adjacent, but the two vertices of degree 3 are not adjacent in H. [2 marks]

[Other reasons are acceptable]

(c) Clearly, 10,000 does not have the sum of its digits equal to 13, so it suffices to search among the set of integers from 1 to 9,999. All such integers can be uniquely expressed in the form $(x_1 x_2 x_3 x_4)_{10}$

[12]

where $0 \le x_i \le 9$ such that

$$x_1 + x_2 + x_3 + x_4 = 13$$ [1 mark]

Thus, we have to find the coefficient of x^{13} in the expansion of $(1+x+x^2+\cdots+x^9)^4$:

$$(1+x+x^2+\cdots+x^9)^4 = \left(\frac{1-x^{10}}{1-x}\right)^4$$

$$= (1-x^{10})^4(1-x)^{-4} = (1-4x^{10}+\cdots)\sum_{n=0}^{\infty}\binom{n+3}{3}x^n$$

[1 mark]

We put $n = 13, 3$

we get $\binom{16}{3} - 4\binom{6}{3} =$

$$\frac{16\times \cancel{15}^{5}\times \cancel{14}^{7}}{\cancel{3}\times\cancel{2}\times 1} - \frac{4\times\cancel{6}\times 5\times 4}{\cancel{3\times 2\times 1}} =$$

$$= 80\times 7 - 80 = 80\times 6 = 480$$

[1 mark]

Answer question 5

(a) The statement is true. [$\frac{1}{2}$ mark]

For, we have $\deg_G(v)$ is even for every vertex v of G. As $\deg_{\bar{G}}(v) = (n-1) - \deg_G(v)$, then

[13] $\deg_{\bar{G}}(v)$ is even for every vertex of $\bar{G}$, hence $\bar{G}$ is Eulerian. [1/2 mark].

(b) The statement is true. This is [1/2 mark] Dirac's Theorem

We may prove it, by using Ore's Theorem:

Let $u, v \in V(G)$, be any two non-adjacent vertices, then $\deg(u) + \deg(v) \geq n/2 + n/2 = n$.

[1/2 mark]

(c) The statement is true. [1/2 mark]

The proof is by induction on "n". If $n=3$

$$(k-1)^3 + (-1)^3 (k-1) = k(k-1)(k-2),$$

which is chromatic polynomial of $C_3 = K_3$.

Let the statement is true at $n-1$:

consider C_n by the deleting and contracting, theorem, we get

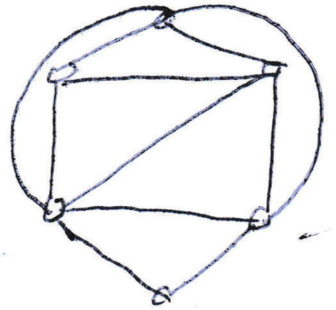
$$P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k)$$

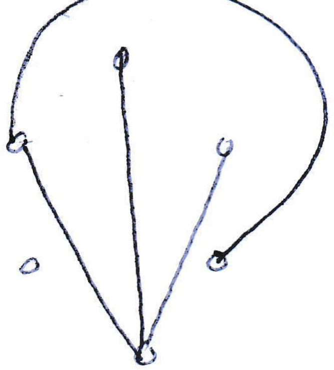
$$= k(k-1)^{n-1} - \left[(k-1)^{n-1} + (-1)^{n-1}(k-1)\right]$$

$$= k(k-1)^{n-1} - (k-1)^{n-1} + (-1)^{n}(k-1)$$

$$= (k-1)^{n} + (-1)^{n}(k-1)$$ [1/2 mark]

(d) The statement is false. [1/2 mark] For example, if we take G the following graph

G  , then both G and $\overline{G}$ [1/2 mark]

are planar. ($\overline{G}$ )

(e) The statement is false. [1/2 mark]

Using the Inclusion-Exclusion principle, Let A be the set of numbers that are Square ~~numbers~~ from 1 to 10^6 $\Rightarrow |A| = \sqrt{10^6} = 1000$

and B be set of the numbers that are cube ~~numbers~~ from 1 to 10^6, then $|B| = \sqrt[3]{10^6} = 100$.

Now $A \cap B$ be the set of numbers that are both square and cube $\Rightarrow$

$$|A \cap B| = \sqrt[6]{10^6} = 10$$

Hence $|A \cup B| = |A| + |B| - |A \cap B|$

$$= 1000 + 100 - 10 = 1090$$

[1/2 mark]

(f) The statement is true. [1/2 mark]

Suppose that G is a bipartite Eulerian graph, then $\deg(v)$ is even for every vertex v of G, and G decomposes into cycles. Since G is bipartite, all these cycles must have even length. It follows that $|E(G)|$ is even, (since $|E(G)| = \sum_i |C_i|$, where $|C_i|$ is even for every i)

16

Answer the Extra question

Let a_r be the number of non negative integer solutions of the equation $x+y+z=r$ subject that $x=0,1,2,\cdots$, $y=0,5,10,\cdots$ and $z=0,5,10,\cdots$. Then the generating function of $(a_r)_{n=0}^{\infty}$ is

$$g(x)=(1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)^2 \quad \text{[1 mark]}$$

$$=\frac{1}{1-x}\cdot\frac{1}{(1-x^5)^2}$$

$$=\sum_{n=0}^{\infty}x^n\sum_{n=0}^{\infty}\binom{n+1}{n}(x^5)^n$$

$$=\sum x^n\sum_{n=0}^{\infty}(n+1)\,x^{5n} \quad \text{[1 mark]}$$

$$=\sum_{n=0}^{\infty}x^n\sum_{n=0}^{\infty}b_n\,x^n, \text{ where}$$

$$b_n=\begin{cases}0 & \text{if } n\not\equiv 0 \pmod 5\\ \frac{n}{5}+1 & \text{if } n\equiv 0 \pmod 5\end{cases}$$

$\boxed{17}$

That is

$$g(x) = \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} b_n x^n$$

$$\sum a_n x^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} b_i \right) x^n$$ [1 mark]

That is, $$a_{100} = \sum_{i=0}^{100} b_i = \sum_{j=0}^{20} b_{5j}$$

$$= \sum_{j=0}^{20} (j+1)$$

$$= \frac{21(22)}{2} = 21(11)$$

$$= 231$$ (1 mark)