Ministry of Education
Al-Imam Mohammad Ibn Saud Islamic University
College of Science
Department of Mathematics and Statistics

Course Name: Combin. and Graphs
Course Code: MAT 354
Semester/Year: First/1437-1438H
Date/Time: 18-04-1438H / 8:00 am
Duration: 2 Hours

Instructions: Only ordinary calculators are allowed.

Final Exam

Model Answer

16\01\2017

Name	ID	section

Q1		**8**
Q2		**6**
Q3		**8**
Q4		**6**
Q5		**12**
Total		**40**

Question 1:

a) **(3pts)** A test contains 100 true/false questions. How many ways a student can answer the questions on the test, if the answer may be left blank?

Solution

We have three possibility for each equation: true, false, or blank. $\left[1 \text{ mark}\right]$. So, the number of ways is 3^{100} $\left[2 \text{ marks}\right]$

b) **(2pts)** How many people are needed to guarantee that at least two were born on the same day of the week?

Solution

The minimum number of persons needed to ensure that at least two people born on the same day of the week is the smallest integer N such that $\left\lceil \frac{N}{7} \right\rceil = 2$ The smallest such integer is N = (2-1) .7 + 1 = 8. $\left[2 \text{ marks}\right]$

c) **(3pts)** How many six characters password can be made from the 10 characters 1, 2, 3, 4, A, B, C, D, E,F, if the password contains at least one digit?

Solution

Let P_6 denote the number of possible passwords. It is easier to find the number of passwords(strings) of (the uppercase letters and digits) that are six characters long, and find those passwords with no digits. Then subtract the later number from the first number. By the product rule, the number of strings of six characters is 10^6 $\left[1 \text{ mark}\right]$ and the number of strings with no digits is 6^6 $\left[1 \text{ mark}\right]$. Hence, $P_6 = 10^6 - 6^6 = 953344.$ $\left[1 \text{ mark}\right]$

Question 2 :

a) **(4pts)** Prove the identity

$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ where n, k and r are nonnegative integers with $r \le n$ *and* $k \le r$.

Solution

By straightforward algebraic calculation, we have

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!}\text{[1 mark]}$$
$$= \frac{n!}{k!(n-r)!(r-k)!}\text{[1 mark]}$$
$$= \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-r)!}\text{[1 mark]}$$
$$= \binom{n}{k}\binom{n-k}{r-k}.\text{[1 mark]}$$

(A combinatorial solution is also acceptable)

b) **(2pts)** A man has 10 identical toys to distribute among three distinct children. How many ways he distributes the toys such that each child receives at least 3 toys and not more than 5 toys?

Solution

Because each child receives at least three but no more than five toys, for each child there is a factor equal to $(x^3+x^4+x^5)$ in the generating function for the sequence c_n, where c_n is the number of ways to distribute n toys. Because there are three children, this generating function is We need the coefficient of x^{10} in this product $(x^3+x^4+x^5)^3$:

$$(x^3+x^4+x^5)^3 =$$
$$= x^9(1+x+x^2)^3$$
$$= x^9\frac{(1-x^3)^3}{(1-x)^3} = x^9(1-x^3)^3(1-x)^{-3}\text{[1 mark]}$$
$$= x^9(1-x^3)^3\sum_0^{\infty}\binom{2+r}{r}x^r$$

So, take $r=1$ and the number of ways is equal $\binom{3}{1} = 3.$[1 mark].

(We may consider this question as the number of integer solutions of the equation $x_1+x_2+x_3=10$ subject to $3\le x_i\le 5$, where x_i is the number of toys received by the child number i, $1\le i\le 3$.)

Question 3:

a) **(4pts)** Use generating functions to solve the recurrence relation $a_n = ٢a_{n-١} + ٣a_{n-٢}$ with $a_٠ = ٧, a_١ = ١$.

Solution

Let g(x) be the generating function for the sequence a_n , that is, g(x) = $\sum_{n=0}^{\infty} a_n x^n$.

We multiply both sides of the given recurrence relation by x^n to obtain

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n$$

We sum both sides of the last equation starting with $n = 2$ to find that

$$\begin{aligned}\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty}(2a_{n-1}x^n + 3a_{n-2}x^n) \\ &= 2\sum_{n=2}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n \\ &= 2x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 3x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} \left[1 \text{ mark}\right]\end{aligned}$$

That is

$$g(x) - a_o - a_1 x = 2x(g(x) - a_0) + 3x^2 g(x)$$

$$(1 - 2x - 3x^2)g(x) = 7 + x - 14x$$

$$g(x) = \frac{7 - 13x}{1 - 2x - 3x^2}. \left[1 \text{ mark}\right]$$

Now, using the partial fractions we get

$$\frac{7 - 13x}{1 - 2x - 3x^2} = \frac{A}{1 + x} + \frac{B}{1 - 3x} = \frac{A(1 - 3x) + B(1 + x)}{(1 + x)(1 - 3x)}$$

and get $A(1 - 3x) + B(1 + x) = 7 - 13x$. Then

$A = 5 \quad and\ B = 2.$ $\left[1 \text{ mark}\right]$ So

$$g(x) = \frac{5}{1 + x} + \frac{2}{1 - 3x} = 5\sum_{n=0}^{\infty}(-1)^n x^n + 2\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}(5(-1)^n + 2.3^n)x^n$$

and $a_n = 5(-1)^n + 7.3^n$. $\left[1 \text{ mark}\right]$

b) (4pts) Solve nonhomogeneous recurrence relation $a_n = ١٠a_{n-١} - ٢٥a_{n-٢} + ٢^n$ together with the initial conditions $a_٠ = ٣ \ \ and\ a_١ = ١٧$.

Solution

This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation are $a_n^{(h)} = c_1 5^n + c_2 n 5^n$ $\left[1 \text{ mark}\right]$, where c_1 and c_1 are constants.

Because F(n) = 2^n , we put a particular solution of the form $a_n^{(p)} = e2^n$, where e is a constant. Substituting the terms of this sequence into the given recurrence relation implies that $e2^n = 10e2^{n-1} - 25e2^{n-2} + 2^n$.Factoring out 2^{n-2} , this equation becomes $4e = 20e - 25e + 4$, which implies that $9e = 4$, or that $e = \frac{4}{9}$ and

$a_n^{(p)} = \frac{4}{9}.2^n.\left[1 \text{ mark}\right]$Then the general solution is $a_n = c_1 5^n + c_2 n5^n + \frac{4}{9}.2^n \left[1 \text{ mark}\right]$. As $a_0 = 3, a_1 = 17$, we get

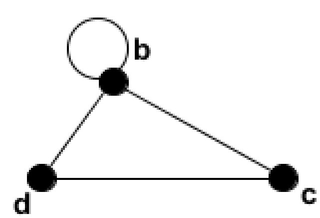
$$c_1 = 3 - \frac{4}{9} = \frac{23}{9} \text{ and } 5c_1 + 5c_2 = 17 - \frac{8}{9} = \frac{145}{9},\ c_2 = \frac{29}{9} - \frac{23}{9} = \frac{6}{9} = \frac{2}{3}$$

and the solution is of the form

$$a_n = \frac{23}{9}5^n + \frac{2}{3}n5^n + \frac{4}{9}.2^n = \frac{1}{9}(5^n(23+6n) + 2^{n+2})\left[1 \text{ mark}\right].$$

Question 4:

Given the following graph G



a) (2pts) Give the adjacency matrix for G.

$$A_G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \left[2 \text{ marks}\right]$$

b) (2pts) How many paths from b to d are of length 3?

We calculate A_G^3:

$$A^2_G = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad A^3_G = \begin{bmatrix} 7 & 5 & 5 \\ 5 & 3 & 4 \\ 5 & 4 & 3 \end{bmatrix} \left[1 \text{ mark}\right]$$

Then there are 5 paths from the vertex b to the vertex d of length 3. $\left[1 \text{ mark}\right]$

c) **(2pts)** Is G Eulerian? Justify your answer.

Yes the graph is Eulerian $\left[1 \text{ mark}\right]$, because the degree of each vertex is even

(deg(b)=4, deg(c)= deg(d)=2. $\left[1 \text{ mark}\right]$

Question 5:

a) **(2pts)** A connected planar simple graph with 150 vertices and 200 edges divides the plane into how many regions?

Solution

From Euler's formula, we have

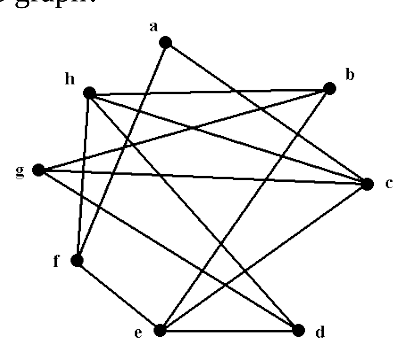
$n + r = m + 2$ $\left[1 \text{ mark}\right]$, where n is the number of vertices, m is the number of edges and r is the number of regions. We have n=150, m=200. Then r=m-n+2=200-150+2=52 regions $\left[1 \text{ mark}\right]$

b) **(2pts)** For which value of n the complete graph K_n is Eulerian?

Solution

The degree of every vertex of the complete graph K_n is n-1. $\left[1 \text{ mark}\right]$ So, the degree of every vertex of K_n is even if and only if n is odd. Hence K_n is Eulerian iff n is odd $\left[1 \text{ mark}\right]$

c) **(4pts)** Is the graph drawn below planar? If it is planar, redraw it without edges crossing, if no explain why? What is the chromatic number of this graph?



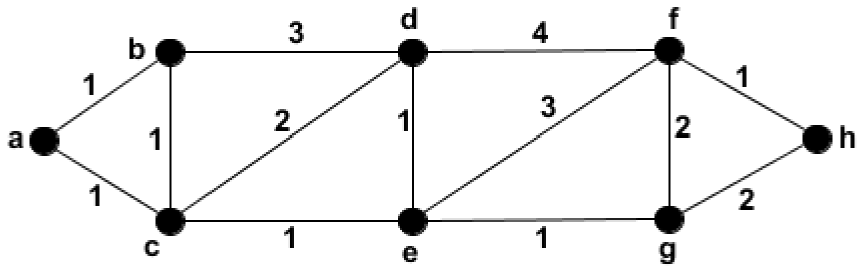
Solution

The subgraph H of the given graph obtained by deleting the vertices a and f and all edges incident with them is isomorphic to $K_{3,3}$. $\left[1 \text{ mark}\right]$. So using Kuratowski's Theorem, the given graph is nonplanar. $\left[1 \text{ mark}\right]$

[Another way: This graph contains no triangles (the graph is bipartite) and have m=13 edges and n=8 vertices. Since m=13 > 2n-4=12, then the graph is nonplanar]

We may color the vertices b, c, d and f with one color and the other vertice by another color. So the chromatic number of the given graph is 2. $\left[2 \text{ marks}\right]$

d) **(4pts)** find the shortest path and its length between **a** and **h** in the given weighted graph.



Using Dijkstra's algorithm

S	a	b	c	d	e	f	g	h
√a	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞
b	$\boxed{0}$	$\boxed{1}$	1	∞	∞	∞	∞	∞
√c	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	4	∞	∞	∞	∞
√e	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	3	$\boxed{2}$	∞	∞	∞ $\left[2 \text{ marks}\right]$
d	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\boxed{3}$	$\boxed{2}$	5	3	∞
√g	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\boxed{3}$	$\boxed{2}$	5	$\boxed{3}$	6
f	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\boxed{3}$	$\boxed{2}$	$\boxed{5}$	$\boxed{3}$	5
√h	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\boxed{3}$	$\boxed{2}$	$\boxed{5}$	$\boxed{3}$	$\boxed{5}$

Hence the length of the shortest path is 5 and the this path is:

a, c, e, g, h. $\left[2 \text{ marks}\right]$

TABLE 1 Useful Generating Functions.	
$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$