Ministry of Education
Al-Imam Mohammad Ibn Saud Islamic University
College of Science
Department of Mathematics and Statistics

Course Name: Combin. and Graphs
Course Code: MAT 354
Semester/Year: First/1438-1439H
Date/Time: 14-04-1439H / 8:00 am
Duration: 2 Hours

Instructions: Only ordinary calculators are allowed.

Final Exam

1\01\2018

Name	ID	section

(Model Answer)

Q1		8
Q2		8
Q3		6
Q4		10
Q5		8
Total		40

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Question 1: (8=4+2+2 pts)

a) How many strings of length four from the letters A,B,C,D,X, Y, and Z are there

i) if letters can be repeated?

7 7 7 7   The number is $7^4 = 2401$ $\boxed{1}$

ii) if no letter can be repeated?

7 6 5 4   The number is $P(7,4) = 7 \times 6 \times 5 \times 4$
$= 840$ $\boxed{1}$

iii) if each string contain the letter A?

The number of strings that do not contain the letter A is 6^4. So, the number is $7^4 - 6^4 =$ $\boxed{1}$

iv) if each string contain the letters A and B?

Let X be the set of strings that do not contain the letter A;
and Y " " " " " " " " " " B.
So, we need to calculate $|X^c \cap Y^c|$. Then $|X^c \cap Y^c| = U - |X \cup Y|$ $\boxed{1}$
$= 7^4 - 6^4 - 6^4 + 5^4$

b) Find the coefficient of x^{19} in the expansion of $(2x^3 - 3x)^9$.

$$(2x^3 - 3x)^9 = \sum_{k=0}^{9} \binom{9}{k} (2x^3)^{9-k} (-3x)^k = \sum \binom{9}{k} 2^{9-k} (-3)^k (x^3)^{9-k} x^k$$

$$\boxed{1} = \sum \binom{9}{k} (-3)^k 2^{9-k} x^{27-2k}$$

. To obtain the coeff of x^{19}, put $27 - 2k = 19 \Rightarrow k = 4 \Rightarrow$ The coeff is $\binom{9}{4} 2^5 (-3)^4$. $\boxed{1}$

c) A basket of fruits has 10 apples and 12 bananas . What is the smallest number of these fruits that should take out at random in the dark from the basket to guarantee that you have at least five pieces of the same fruit? Let N be the required number.

There are two kinds of fruits: these are the boxes,
So, we need to solve

$$\left\lceil \frac{N}{2} \right\rceil = 5 \;\boxed{1} \Rightarrow N = 2(5-1)+1 = 9 \;\boxed{1}$$

Question 2:(8=4+4 pts)

a) Solve the recurrence relation $a_n = 7a_{n-1} - 6a_{n-2}$ together with initial conditions $a_0 = 1$, $a_1 = 2$.

The characteristic equation of the recurrence relation is $r^2 - 7r + 6 = 0$. Its roots are $r = 1$ and $r = 6$. $\boxed{1}$

So, the solution can be written as $a_n = c_1 + c_2 \cdot 6^n$, where $\boxed{1}$ c_1 and c_2 are constants. From the initial conditions, it follows that

$$c_1 + c_2 = 1, \quad \boxed{1}$$
$$c_1 + 6c_2 = 2.$$

Solving these two equations shows that $c_1 = \frac{4}{5}$, $c_2 = \frac{1}{5}$

Hence the solution is $a_n = \frac{4}{5} + \frac{1}{5} \cdot 6^n$. $\boxed{1}$

b) Let n and k be a positive integers with $k \leq n$. Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Use this identity to simplify the following:

$$\binom{10}{5} + \binom{10}{6} + \binom{11}{7} + \binom{12}{8} + \cdots + \binom{19}{15} + \binom{20}{16}$$

RHS $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!(k-1)!}\left[\frac{1}{n-k+1} + \frac{1}{k}\right]$ $\boxed{1}$

$= \frac{n!}{(n-k)!(k-1)!}\left[\frac{k+n-k+1}{(n-k+1)k}\right] = \frac{n!\,(n+1)}{(n-k)!(k-1)!\,(n-k+1)\,k}$ $\boxed{1}$

$= \frac{(n+1)!}{(n-k+1)!\,k!} = \binom{n+1}{k} = $ LHS

Using this identity, we get that the sum

is equal to $\binom{21}{16}$. $\boxed{2}$ See the page behind

$$\underline{\binom{10}{5}+\binom{10}{6}}+\binom{11}{7}+\cdots+\binom{20}{16}=$$

$$=\underline{\binom{11}{6}+\binom{11}{7}}+\binom{12}{8}+\cdots+\binom{20}{16}=$$

$$=\underline{\binom{12}{7}+\binom{12}{8}}+\binom{13}{9}+\cdots+\binom{20}{16}=$$

$$=\underline{\binom{13}{8}+\binom{13}{9}}+\cdots+\binom{20}{16}$$

$$\vdots$$

$$=\binom{20}{15}+\binom{20}{16}=\binom{21}{16}.$$

Question 3: :(6=4+2 pts)

a) Use generating functions to solve the recurrence relation $a_n = 2a_{n-1} + 3.2^n$ together with the initial condition $a_0 = 3$.

Let $g(x)$ be the generating function for $\{a_n\}_{n=0}^{\infty}$.
Multiply the given recurrence relation by x^n, we get

$a_n x^n = 2a_{n-1}x^n + 3.2^n x^n$. $\boxed{1}$ Taking the summation from 1 to ∞, we get

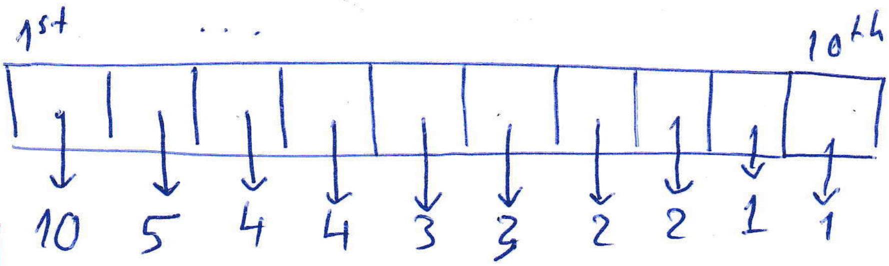
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1}x^n + \sum_{n=1}^{\infty} 3.2^n x^n \quad \boxed{1}$$

$$\Rightarrow g(x) - a_0 = 2x\,g(x) + \cancel{6}x \cdot \frac{1}{1-2x} \Rightarrow (1-2x)g(x) = 3 + \frac{\cancel{6}x}{1-2x}$$

$$(1-2x)g(x) = \frac{3(1-2x)+\cancel{6}x}{1-2x} = \frac{3}{1-2x} \Rightarrow g(x) = \frac{3}{(1-2x)^2} \quad \boxed{1}$$

$$= 3\sum \binom{n+1}{n}(2x)^n \Rightarrow g(x) = 3(n+1).2^n \quad \boxed{1}$$

b) There are 5 distinct white cars and 5 distinct black cars must be park in a line. How many ways are there to park these cars so that white and black cars alternate?



$\boxed{2}$

The number is $2(5!)^2$. Or, we have: Two patterns

$\boxed{1}$	w	b	w	b	w	b	w	b	w	b
$\boxed{1}$	b	w	b	w	b	w	b	w	b	w

done by $(5!)^2$

" " $(5!)^2$

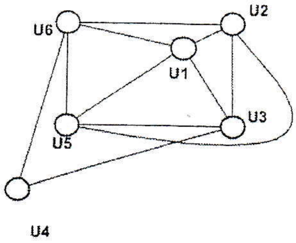
So, the required number $(5!)^2 + (5!)^2 = 2(5!)^2$

Question 4: (**10=2+2+2+2+2 pts**)

A graph G is given below:

1) Find the adjacency matrix of G

Let A be the adjacency matrix of G w.r.t. the list $u_1, u_2, \dots, u_6$



$\therefore$

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2

2) Find an Euler circuit in G.

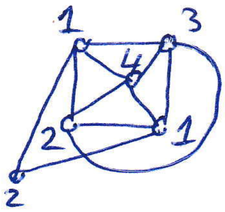
$u_1, u_3, u_5, u_1, u_6, u_2, u_5, u_6, u_4, u_3, u_2, u_1$ $\boxed{2}$

3) Find a Hamilton circuit in G.

$u_1, u_2, u_3, u_4, u_6, u_5, u_1$ $\boxed{2}$

4) Find the chromatic number $\chi(G)$.

$K_4 \subseteq G$ $\Rightarrow \chi(G) \geq 4$, $\chi(G) = 4$ $\boxed{2}$



5) Is G planar? Justify your answer. $\boxed{\text{NO}}$ $\boxed{1}$

G is non planar,

because G contains a subgraph homeomorphic to K_5. $\boxed{1}$

Question ~~4~~ 5 (8=2+2+4 **pts**)

a) Can a simple graph exist with 3 vertices of degree 3 and four vertices of degree 5? Justify your answer.

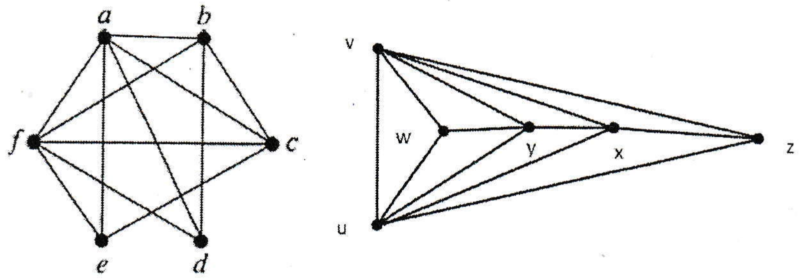
$\boxed{1}$ $\boxed{\text{No}}$, because any undirected graph must

$\boxed{1}$ have an even number of vertices of odd degree.

OR

$$\sum_{v \in G} \deg(v) = 3 \times 3 + 4 \times 5 = 29 \neq 2|E|$$

b) Find an isomorphism between the graphs given below.



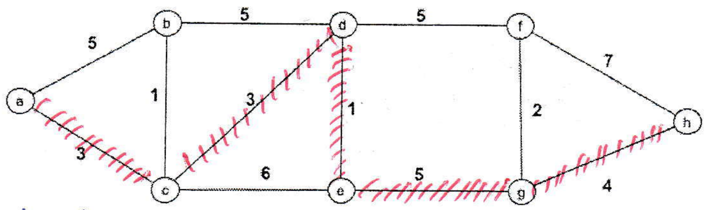
G H

Define

$g: V(G) \to V(H)$ as follows $\boxed{2}$

$g(a)=v$, $g(f)=u$, $g(b)=x$, $g(c)=y$, $g(d)=z$ and $g(e)=w$.

c) Use Dijkstra's algorithm to find a shortest part from the vertex **a** to the vertex **h**. What is length of this path?



Using Dijkstra's algorithm,

Marked	a	b	c	d	e	f	g	h
a	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞
c	$\boxed{0}$	5	$\boxed{3}$	∞	∞	∞	∞	∞
b	$\boxed{0}$	$\boxed{4}$	$\boxed{3}$	6	9	∞	∞	∞
d	$\boxed{0}$	$\boxed{4}$	$\boxed{3}$	$\boxed{6}$	9	∞	∞	∞
e	$\boxed{0}$	$\boxed{4}$	$\boxed{3}$	$\boxed{6}$	$\boxed{7}$	11	∞	∞
f	$\boxed{0}$	$\boxed{4}$	$\boxed{3}$	$\boxed{6}$	$\boxed{7}$	$\boxed{11}$	12	∞
g	$\boxed{0}$	$\boxed{4}$	$\boxed{3}$	$\boxed{6}$	$\boxed{7}$	$\boxed{11}$	$\boxed{12}$	18
	$\boxed{0}$	$\boxed{4}$	$\boxed{3}$	$\boxed{6}$	$\boxed{7}$	$\boxed{11}$	$\boxed{12}$	$\boxed{16}$

$\boxed{2}$

So, the length of the shortest path from a to h is equa[l] $\boxed{1}$ 16. The path,

$\boxed{1}$ a, c, d, e, g, h.

as described above