[1]

Key Answer of Final

Exam of MAT 354

2^{nd} Semester 2016/2017

Answer question (1)

(a) Let A be the set of all positive integers from 1000 to 5000 that are divisible by 9.

$$\Rightarrow |A| = \left\lfloor \frac{5000}{9} \right\rfloor - \left\lfloor \frac{999}{9} \right\rfloor = 555 - 111 = 444$$

and

let B be the set of integers from 1000 to 5000 that are divisible by 11

$$\Rightarrow |B| = \left\lfloor \frac{5000}{11} \right\rfloor - \left\lfloor \frac{999}{11} \right\rfloor = 454 - 90 = 364$$

and $$|A \cap B| = \left\lfloor \frac{5000}{99} \right\rfloor - \left\lfloor \frac{999}{99} \right\rfloor = 50 - 10 = 40$$

[1 mark]

That is, the number of positive integers from 1000 to 5000 that are divisible by 9 or 11 is given by the Inclusion-Exclusion principle:

(2)

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 444 + 364 - 40 = 768. \text{ [1 mark]}$$

Thus, the number of integers from 1000 to 5000 that are not divisible by 9 and not divisible by 11 is equal

$$4001 - 768 = 3233 \text{ [1 mark]}$$

(b) Let N be the minimum number of students in the school that gurantee that at least 6 were born in the same month, then

$$\left\lceil \frac{N}{12} \right\rceil = 6 \Rightarrow N = 12(6-1)+1$$
$$= 61. \text{ [2 marks]}$$

(c) $$\left(x^2 + \frac{2}{x^2}\right)^{80} = \sum_{i=0}^{80} \binom{80}{i} (x^2)^{80-i} \left(\frac{2}{x^2}\right)^i$$

$$= \sum_{i=0}^{80} \binom{80}{i} x^{160-2i} \cdot 2^i x^{-2i}$$

$$= \sum_{i=0}^{80} 2^i \binom{80}{i} x^{160-4i} \quad \text{[2 marks]}$$

$$160 - 4i = 20 \Rightarrow 4i = 140$$
$$\Rightarrow i = \frac{140}{4} = 35$$

$\Rightarrow$ The coefficient of x^{20} is equal

$$2^{35}\binom{80}{35}. \quad [1 \text{ mark}]$$

Answer question (2)

(a) LHS $\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1}$, (Pascal's Id.) [1 mark]

$$= \frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right], \quad [1 \text{ mark}]$$

$$= \frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n}\right], \quad \left(\text{as} \binom{n}{r} = \binom{n}{n-r}\right) \quad [1 \text{ mark}]$$

$$= \frac{1}{2}\binom{2n+2}{n+1}, \quad (\text{Pascal's Id}) \quad [1 \text{ mark}]$$

(*) Any other solution is acceptable.

(b) There are $\binom{16}{8}$ ways to choose the [1 mark] committee to be composed only from men, $\binom{16}{7}\binom{12}{1}$ ways if there are 7 men and one [1 mark] women, $\binom{16}{6}\binom{12}{2}$ ways if there are 6 men and

(4) two women, $\binom{16}{5}\binom{12}{3}$ ways if there are [1 mark]
5 men and 3 women. Thus the total number
is equal.

$$\binom{16}{8} + \binom{16}{7}\binom{12}{1} + \binom{16}{6}\binom{12}{2} + \binom{16}{5}\binom{12}{3}$$
[1 mark]

Answer question (3)

(a) Let $g(x)$ be a generating function of the sequence $(a_n)_{n=0}^{\infty}$. That is $g(x) = \sum_{n=0}^{\infty} a_n x^n$,

multiply the given recurrence relation by x^n, we get

$$a_n x^n = 3a_{n-1}x^n + 2^{n-1}x^n$$ [1 mark]

Taking the summation from $n=1$ to ∞,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1}x^n + \sum_{n=1}^{\infty} 2^{n-1}x^n$$

$$\Rightarrow \quad g(x) - a_0 = 3x\,g(x) + \frac{x}{1-2x}$$

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$$g(x) - 3x\,g(x) = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}$$

$$g(x) = \frac{1-x}{(1-2x)(1-3x)}.$$ [1 mark]

Using the partial fractions

$$\frac{1-x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x} \Rightarrow$$

$$A(1-3x) + B(1-2x) = 1-x$$

At $x = 1/2 \Rightarrow A\left(1-\frac{3}{2}\right) = 1-\frac{1}{2}$

$\Rightarrow -\frac{1}{2}A = \frac{1}{2} \Rightarrow A = -1$

At $x = \frac{1}{3} \Rightarrow B\left(1-\frac{2}{3}\right) = 1-\frac{1}{3}$

$\Rightarrow B\left(\frac{1}{3}\right) = \frac{2}{3} \Rightarrow B = 2$

[1 mark]

$$\Rightarrow g(x) = \frac{-1}{1-2x} + \frac{2}{1-3x} = -\sum_{n=0}^{\infty} 2^n x^n + 2\sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} (-2^n + 3^n \cdot 2)\, x^n$$

$$\Rightarrow a_n = -2^n + 2 \cdot 3^n = 2 \cdot 3^n - 2^n$$

[1 mark]

[6] (b) The characteristic equation

$$r^2 - 7r + 12 = 0$$

$$(r-3)(r-4) = 0$$

Thus, we have two distinct char. roots

$r_1 = 3$, $r_2 = 4 \Rightarrow$

$$a_n = c_1 \cdot 3^n + c_2 \cdot 4^n$$ [2 marks]

As $a_0 = 0 \Rightarrow c_1 + c_2 = 0$

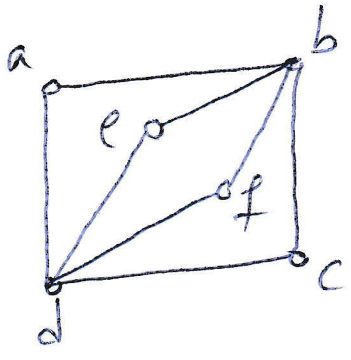
As $a_1 = 2 \Rightarrow 3c_1 + 4c_2 = 2 \Rightarrow c_2 = 2$

$\Rightarrow c_1 = -2$

$$a_n = (-2) \cdot 3^n + 2 \cdot 4^n$$ [2 marks]

Answer question (4)

G



Let A_G be the adjacency matrix with respect to the list a, b, c, d, e, f of the vertices given in the above figure.

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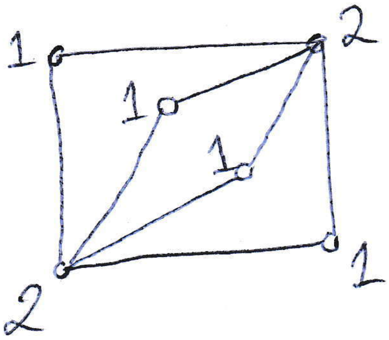
(a)

$$A_G = \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \left[\begin{array}{cccccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

[2 marks]

(b)



$\Rightarrow \chi(G) = 2$ [2 marks]

(c) As the degree of every vertex of G is even, then G has an Euler circuit. [1 mark]

a, b, c, d, e, b, f, d, a is an Euler circuit. [1 mark]

(d) This graph has no Hamilton circuit. If it did, then certainly the circuit would have to contain edges ab and ad, since these are the only edges incident to vertex a. By the

[8] Same reasoning, the circuit would have to contain edges bc and cd. These 4 edges already complete a circuit, and this circuit omit the two vertices on the inside. Therefore there is no Hamilton circuit. [2 marks]

Answer ques tion (5)

(a) $G \not\cong H$, <u>because</u>, the vertex of degree 2 in G is adjacent to two non adjacent vertices, <u>but</u> the vertex of degree 2 in H is adjacent to two adjacent vertices. 2 marks

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(b) In $K_{m,n}$, we have n vertices each of degree m and m vertices each of degree n. Thus $K_{m,n}$ is Eulerian $\Longleftrightarrow$ $\deg(v)$ is even $\forall v \in V(K_{m,n})$

[2 marks] $\Longleftrightarrow$ m and n are even

Using Dijkstra's Algorithm,

	a	b	c	d	e	f	g	h	i	j	z
a	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
c	$\boxed{0}$	2	$\boxed{1}$	∞	∞	∞	∞	∞	∞	∞	∞
b	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	4	∞	3	∞	∞	∞	∞	∞
f	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	4	4	$\boxed{3}$	∞	∞	∞	∞	∞
d	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	4	$\boxed{3}$	5	6	∞	∞	∞
e	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	$\boxed{4}$	$\boxed{3}$	9	6	∞	∞	∞
h	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	$\boxed{4}$	$\boxed{3}$	9	$\boxed{6}$	∞	∞	∞
g	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	$\boxed{4}$	$\boxed{3}$	$\boxed{8}$	$\boxed{6}$	∞	9	∞
i	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	$\boxed{4}$	$\boxed{3}$	$\boxed{8}$	$\boxed{6}$	$\boxed{9}$	9	∞
j	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	$\boxed{4}$	$\boxed{3}$	$\boxed{8}$	$\boxed{6}$	$\boxed{9}$	$\boxed{9}$	12
z	$\boxed{0}$	$\boxed{2}$	$\boxed{1}$	$\boxed{4}$	$\boxed{4}$	$\boxed{3}$	$\boxed{8}$	$\boxed{6}$	$\boxed{9}$	$\boxed{9}$	$\boxed{12}$

The shortest path is [2 marks]

a, c, f, h, g, i, z and its

length equal to 12. [2 marks]