

Kingdom of Saudi Arabia

Ministry of Education

Al-Imam Mohammed Ibn Saud Islamic University

--- College of Science ---

Department of mathematics & statistics



المملكة العربية السعودية

وزارة التعليم

جامعة الإمام محمد بن سعود الإسلامية

--- كلية العلوم ---

قسم الرياضيات و الاحصاء

**FINAL EXAMINATION**

MAT 102

CALCULUS 2

Semester 2- 2017/2018 (1438/1439 Heg.)

STUDENT NAME: \_\_\_\_\_ STUDENT ID: \_\_\_\_\_ DURATION: 2 HOURS

Question:	1	2	3	4	Total
Points:	12	6	14	8	40
Score:					

No calculators allowed. Answers written outside the allocated space will NOT be graded!!!

1. 12 points

1. Evaluate the following integrals

(a)  $\int_0^1 x \tan^{-1} x \, dx$

(3pts)

Using integration by parts, let

$u = \tan^{-1} x, \quad dv = x \, dx$   
 $\Rightarrow du = \frac{1}{1+x^2} dx, \quad v = \frac{x^2}{2}$

Hence,  $\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$   
 $= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left( \frac{1+x^2-1}{1+x^2} \right) dx$   
 $= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \int \frac{1}{1+x^2} dx$   
 $= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$

Hence  $\int_0^1 x \tan^{-1} x \, dx = \left[ \left( \frac{x^2}{2} + \frac{1}{2} \right) \tan^{-1} x - \frac{1}{2} x \right]_0^1$   
 $= \left[ \left( \tan^{-1} 1 + \frac{1}{2} \right) - \left( \left( 0 + \frac{1}{2} \right) \tan^{-1} 0 - \frac{1}{2} \cdot 0 \right) \right] = \frac{\pi}{4} - \frac{1}{2}$

(1)

(b)  $\int \frac{2}{x^3 + 2x^2 + 2x} dx$

(2pts)

Using the partial fractions,

$$\frac{2}{x^3 + 2x^2 + 2x} = \frac{2}{x(x^2 + 2x + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 2}$$

(since  $x^2 + 2x + 2$  has no real factorization)

multiplying both sides by  $x(x^2 + 2x + 2)$ , we get

$$2 = A(x^2 + 2x + 2) + x(Bx + C)$$

We match up the coefficients of like powers of  $x$ :

$$A + B = 0, \quad 2A + C = 0, \quad \text{and} \quad 2A = 2. \quad \text{We get}$$

$$A = 1, \quad B = -1, \quad C = -2. \quad \text{Hence} \quad \boxed{0.5}$$

$$\int \frac{2}{x(x^2 + 2x + 2)} dx = \int \left( \frac{2}{x} - \frac{(x+2)}{x^2 + 2x + 2} \right) dx$$

$$= 2 \ln|x| - \frac{1}{2} \int \frac{(2x + 2 + 2)}{x^2 + 2x + 2} dx \quad \boxed{0.5}$$

$$= 2 \ln|x| - \frac{1}{2} \ln|x^2 + 2x + 2| - \tan^{-1}(x+1)$$

(As  $x^2 + 2x + 2 = (x+1)^2 + 1$ )  $\boxed{1}$

(c)  $\int_{-\pi/3}^0 \sqrt{\cos x} \sin^3 x dx$

(2pts)

Let  $u = \cos x \Rightarrow du = -\sin x dx$   $\boxed{0.5}$

As  $x = -\pi/3 \Rightarrow u = 1/2$ , and as  $x = 0 \Rightarrow u = 1$ .

$$\int_{-\pi/3}^0 \sqrt{\cos x} \sin^3 x dx = \int_{1/2}^1 \sqrt{u} (1 - u^2) \sin x dx$$

$$= \int_{1/2}^1 \sqrt{u} (1 - u^2) (-du) = \int_{1/2}^1 (u^{5/2} - u^{3/2}) du$$

$$= \left[ \frac{2}{7} u^{7/2} - \frac{2}{3} u^{3/2} \right]_{1/2}^1 \quad \boxed{1}$$

$$= \left( \frac{2}{7} - \frac{2}{3} \right) - \left( \frac{2}{7} \left( \frac{1}{2} \right)^{7/2} - \frac{2}{3} \left( \frac{1}{2} \right)^{3/2} \right)$$

$$= -\frac{8}{21} - \left( \frac{\sqrt{2}}{56} - \frac{\sqrt{2}}{6} \right) \quad \boxed{0.5}$$

$$= -\frac{8}{21} + \frac{25\sqrt{2}}{168}$$

(2)



$$(d) \int \frac{x+2}{\sqrt{4-x^2}} dx$$

(2pts)

$$I = \int \frac{x+2}{\sqrt{4-x^2}} dx = \int \frac{x}{\sqrt{4-x^2}} dx + 2 \int \frac{dx}{\sqrt{4-x^2}}$$

$$= -\sqrt{4-x^2} + 2 \int \frac{dx}{\sqrt{4-x^2}} \quad [0.5]$$

put  $x = 2 \sin \theta$ , for  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ , we get  $[0.5]$

$$I = -\sqrt{4-x^2} + 2 \int \frac{2 \cos \theta d\theta}{\sqrt{4-4\sin^2 \theta}} = -\sqrt{4-x^2} - 2 \int \frac{2 \cos \theta}{2 \cos \theta} d\theta$$

$$= -\sqrt{4-x^2} - 2 \int d\theta = -\sqrt{4-x^2} - 2\theta$$

$$= -\sqrt{4-x^2} - 2 \sin^{-1} \left( \frac{x}{2} \right) + C$$

2. Determine the convergence or divergence of  $\int_0^{\infty} \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx$

(3pts)

Substitute  $u = \sqrt{x}$

$$\int \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx = \int 2e^{-u} du = -2e^{-u} + C$$

$$= -\frac{2}{e^{\sqrt{x}}} + C \quad [1]$$

$$\text{Hence } \int_0^1 \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx = \lim_{R \rightarrow 0^+} \int_R^1 \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx + \lim_{R \rightarrow \infty} \int_1^R \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx$$

$$= \lim_{R \rightarrow 0^+} \left[ -\frac{2}{e^{\sqrt{x}}} \right]_R^1 + \lim_{R \rightarrow \infty} \left[ -\frac{2}{e^{\sqrt{x}}} \right]_1^R \quad [1]$$

$$= \lim_{R \rightarrow 0^+} \left[ -\frac{2}{e} + \frac{2}{e^{\sqrt{R}}} \right] + \lim_{R \rightarrow \infty} \left[ -\frac{2}{e^{\sqrt{R}}} + \frac{2}{e} \right]$$

$$= \left( -\frac{2}{e} + 2 \right) + \left( 0 + \frac{2}{e} \right) = 2 \quad [1]$$

(3)

2. 6 points

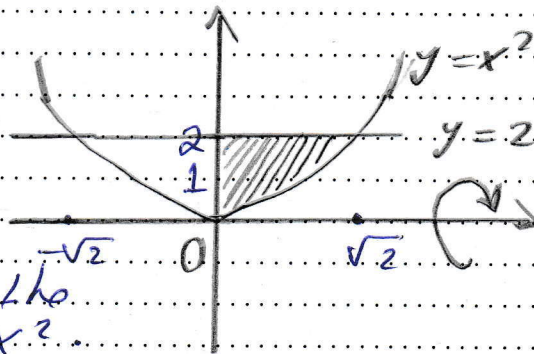
1. Let  $R$  be the region in the first quadrant bounded by the graphs of  $y = x^2$ ,  $x = 0$  and  $y = 2$ . Compute the volume of the solid formed by revolving  $R$  about

(a) the  $x$ -axis and (b) the line  $y = 4$

(3pts)

The line  $y=2$  intersects the curve  $y=x^2$  at  $x=\pm\sqrt{2}$ .

(a) The outer radius,  $r_o$ , is the distance from the  $x$ -axis to the line  $y=2$ . That is  $r_o=2$ . The inner radius,



$r_i$ , is the distance from the  $x$ -axis to the curve  $y=x^2$ .

That is  $r_i = x^2$ . Hence, the required volume

$$\pi \int_0^{\sqrt{2}} (2^2 - (x^2)^2) dx = \pi \int_0^{\sqrt{2}} (4 - x^4) dx \quad [1]$$

$$= \pi \left[ 4x - \frac{x^5}{5} \right]_0^{\sqrt{2}} = \pi (4\sqrt{2} - \frac{(\sqrt{2})^5}{5})$$

$$= \pi \frac{16}{5} = \frac{16}{5} \pi \quad [1]$$

(b)  $r_o = 4 - x^2$ ,

$r_i = 4 - 2 = 2$ , the required volume,

$$\pi \int_0^{\sqrt{2}} [(4-x^2)^2 - 2^2] dx \quad [1]$$

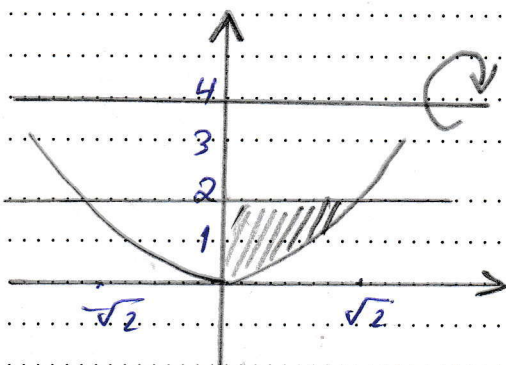
$$= \pi \int_0^{\sqrt{2}} (16 - 8x^2 + x^4 - 4) dx$$

$$= \pi \int_0^{\sqrt{2}} (12 - 8x^2 + x^4) dx = \pi \left[ 12x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^{\sqrt{2}}$$

$$= \pi \left[ 12\sqrt{2} - \frac{8(\sqrt{2})^3}{3} + \frac{(\sqrt{2})^5}{5} \right] = \pi \left( \frac{180 - 80 + 12}{15} \right) \quad [1]$$

$$= \frac{112}{5} \pi$$

(4)





2. Use the method of cylindrical shells to find the volume of the solid formed by revolving the region bounded by the graphs of  $y = x^2$ ,  $y = -x^2$  and  $x = 2$ , about the  $y$ -axis. (3pts)

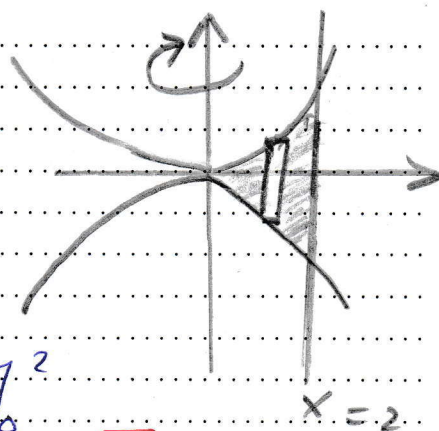
$$\text{radius} = x$$

$$\text{height} = x^2 - (-x^2) = 2x^2$$

$$V = \int_0^2 2\pi x (2x^2) dx \quad [1]$$

$$= 4\pi \int_0^2 x^3 dx = 4\pi \left[ \frac{x^4}{4} \right]_0^2$$

$$= 16\pi. \quad [1]$$



3. 14 points

1. Investigate the convergence or divergence of the following series:

(a)  $\sum_{k=1}^{\infty} \frac{\cos k}{e^k}$

(2pts)

To test for absolute convergence, we consider the series of absolute values,  $\sum_{k=1}^{\infty} \left| \frac{\cos k}{e^k} \right|$ .

$$\text{As } \left| \frac{\cos k}{e^k} \right| = \frac{|\cos k|}{e^k} \leq \frac{1}{e^k}, \quad [1]$$

since  $|\cos k| \leq 1$ , for all  $k$ , and as  $\sum_{k=1}^{\infty} \frac{1}{e^k}$  converges (by using the geometric series, or by using the ratio test)  $\Rightarrow$  By using the comparison test  $\sum_{k=1}^{\infty} \left| \frac{\cos k}{e^k} \right|$  converges too. [0.5]

Consequently, the original series converges absolutely, and hence converges. [0.5]

(5)

$$(b) \sum_{k=1}^{\infty} \left( \frac{k+10}{2k+1} \right)^{2k}$$

Let  $a_k = \left( \frac{k+10}{2k+1} \right)^{2k}$ .

(2pts)

Using the root test, we get

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \sqrt[k]{\left| \frac{k+10}{2k+1} \right|^{2k}} = \lim_{k \rightarrow \infty} \left( \frac{k+10}{2k+1} \right)^2$$

$= \frac{1}{4} < 1$ . Hence the series is absolutely convergent.

$$(c) \sum_{k=1}^{\infty} \frac{k^{10}}{10^k}$$

(2pts)

Let  $a_k = \frac{k^{10}}{10^k}$ . Using the ratio test, we get

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^{10} \cdot 10^k}{10^{k+1} \cdot k^{10}} \right|$$

$$= \frac{1}{10} \lim_{k \rightarrow \infty} \left( \frac{k+1}{k} \right)^{10} = \frac{1}{10} < 1.$$

Hence the given series is absolutely convergent.

2. Determine the radius and interval of convergence of  $\sum_{k=2}^{\infty} \frac{x^k}{4^k \ln k}$

(4pts)

Let  $a_k = \frac{x^k}{4^k \ln k}$ . From the ratio test, we have

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{k+1} \cdot 4^k \ln k}{4^{k+1} \ln(k+1) \cdot x^k} \right|$$

$$= \lim_{k \rightarrow \infty} \frac{1}{4} |x| \frac{\ln(k+1)}{\ln k}$$

$$= \frac{1}{4} |x| \cdot 1 \quad \left( \text{as } \lim_{k \rightarrow \infty} \frac{\ln(k+1)}{\ln k} = 1 \right)$$

That is, the series converges absolutely for  $|x| < 4$  and

diverges for  $|x| > 4$ . It remains only to test the end points of the interval:  $x = \pm 4$ .

Using L'Hopital's rule

$$\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} = \lim_{x \rightarrow \infty} \frac{1/x+1}{1/x}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1$$



For  $x=4$ , we have  $\sum_{k=2}^{\infty} \frac{x^k}{4^k k!} = \sum_{k=2}^{\infty} \frac{1}{k!}$

as  $k > k!$  for every  $k \geq 2$ , we get  $\frac{1}{k} < \frac{1}{k!}$ ,

and as  $\sum \frac{1}{k}$  diverges, then using the comparison test, we get  $\sum_{k=2}^{\infty} \frac{1}{k!}$  diverges too. 0.5

For  $x=-4$ , we have  $\sum_{k=2}^{\infty} \frac{x^k}{4^k k!} = \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}$  0.5

which is convergent using the alternating series test. Hence, the interval of convergence is  $[-4, 4)$  and the radius of convergence is  $r=4$ . 1

3. Find the Taylor series for  $f(x) = \cos x$ , expanded about  $x = \pi/2$  and determine its radius and interval of convergence. (4pts)

$f(x) = \cos x$ ,  $f(\pi/2) = 0$

$f'(x) = -\sin x$ ,  $f'(\pi/2) = -1$

$f''(x) = -\cos x$ ,  $f''(\pi/2) = 0$  1

$f'''(x) = \sin x$ ,  $f'''(\pi/2) = 1$

$f^{(4)}(x) = \cos x$ ,  $f^{(4)}(\pi/2) = 0$ . Therefore  $\cos x =$

$= (x - \pi/2) - \frac{(x - \pi/2)^3}{3!} + \frac{(x - \pi/2)^5}{5!} - \frac{(x - \pi/2)^7}{7!} + \dots$

$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x - \pi/2)^{2k+1}$  1 By the ratio test,

$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} (x - \pi/2)^2 \frac{(2k+1)!}{(2k+3)!} = 0$  1

Hence the interval of convergence is  $(-\infty, \infty)$

and the radius of convergence is  $\infty$ . 1

4. 8 points

1. Consider the parametric curve given by  $x = 2 + 3 \cos t$ ,  $y = 3 + 2 \sin t$

(a) Find an x-y equation for the curve.

(2pts)

put  $\frac{x-2}{3} = \cos t$ ,  $\frac{y-3}{2} = \sin t$ . [1]

As  $\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$

$\Rightarrow \frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$ , which means [1]

that the plane curve lies on the ellipse with center  $(2, 3)$ .

(b) Identify all points at which the curve has a horizontal or vertical tangent line. (2pts)

The horizontal tangent lines occur where

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2 \cos t}{-3 \sin t} = 0$ , which occur only

when  $2 \cos t = 0$ , provided that  $x'(t) = -3 \sin t \neq 0$ . [1]

$\sin \neq \cos t = 0$  only when  $t = (2n+1)\pi/2$ ,  $n \in \mathbb{Z}$ .

The corresponding points on the curve are

$(x(\pi/2), y(\pi/2)) = (2, 5)$ ;  $(x(3\pi/2), y(3\pi/2)) = (2, 1)$ . [1]

For vertical tangent lines, we need  $\sin t = 0$ , provided that  $\cos t \neq 0$ . So  $t = n\pi$ ,  $n \in \mathbb{Z}$ . [1]

(Clearly that  $\cos(n\pi) \neq 0$ ). The corresponding points are

$(x(\pi), y(\pi)) = (-1, 3)$ ,

$(x(2\pi), y(2\pi)) = (5, 3)$ . [1]



2. Find all polar coordinate representation for the rectangular point  $(2, 2\sqrt{3})$ . (2pts)

with  $x=2$  and  $y=2\sqrt{3}$ , we have  $r^2 = x^2 + y^2 = 2^2 + (2\sqrt{3})^2$   
 $= 16 \Rightarrow r = \pm 4$ . Also,  $\tan \theta = y/x = \frac{2\sqrt{3}}{2} = \sqrt{3}$  [1]  
 $\Rightarrow \theta = \pi/3$ . All of the <sup>polar points</sup> must have the form  
 $(4, \pi/3 + 2n\pi)$  or  $(-4, \pi/3 + \pi + 2n\pi)$ , for  
any integer  $n$ . [1]

- -BONUS - -

Identify the center, foci and vertices of the ellipse  $25x^2 + 4y^2 - 100x + 8y + 4 = 0$  (3pts)

$25x^2 - 100x + 4y^2 + 8y = -4 \Rightarrow$   
 $25(x^2 - 4x) + 4(y^2 + 2y) = -4 \Rightarrow$   
 $25(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -4 + 100 + 4$   
 $\Rightarrow 25(x-2)^2 + 4(y+1)^2 = 100$ . Divide by 100  
for both sides, we get  $\frac{(x-2)^2}{4} + \frac{(y+1)^2}{25} = 1$  [1]

Thus, we have  $\frac{(x-2)^2}{2^2} + \frac{(y+1)^2}{5^2} = 1$ .

The center is at  $(2, -1)$ . As  $c^2 = a^2 - b^2 = 21$  [05]

$\Rightarrow c = \sqrt{21}$ . The major axis is parallel to the  $y$ -axis.

The foci are at  $(2, -1 + \sqrt{21})$  and  $(2, -1 - \sqrt{21})$  [1]

The vertices are at  $(2, 4)$  and  $(2, -6)$ . [05]

THE END