Kingdom of Saudi Arabia
Ministry of Education
Al-Imam Mohammed Ibn Saud
Islamic University
College of Science

المملكة العربية السعودية
وزارة التعليم
جامعة الإمام محمد بن سعود الإسلامية
-كليـــة العـــــلوم-

Semester 2 - 1438/1437

Sun. 18/08/1438
Duration: 2H 30Min

Final Exam

CALCULUS II
MAT 106

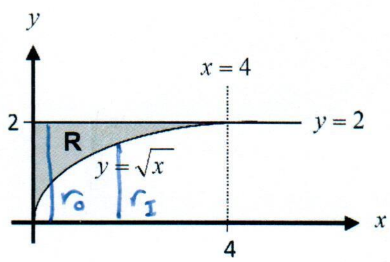
Section:______________ Student Name ______________________________

Answers written outside the allocated space will NOT be graded. **Calculators are not allowed.**

Question:	1	2	3	4	Total
Points:	15	7	6	12	40
Score:					

1. $\boxed{\text{15 points}}$

(a) $\boxed{\text{3 points}}$ Let R be the region bounded by $y = \sqrt{x}$, the y-axis, and the $y = 2$, as shown in the figure. Find the volume of the solid resulting from revolving the region R about the x-axis.



$V = V_o - V_I$ | $r_o = 2$

$= \int_0^4 \pi (r_o)^2\, dx - \int_0^4 \pi (r_I)^2\, dx$ | $r_I = \sqrt{x}$

0.5 0.5

$= \int_0^4 \pi\, 2^2\, dx - \int_0^4 \pi (\sqrt{x})^2\, dx$

0.5 0.5

$= (4\pi x)\Big|_0^4 - \left(\frac{\pi x^2}{2}\right)\Big|_0^4$

$= 16\pi - 8\pi = \boxed{8\pi}$

0.5 0.5

(b) $\boxed{\text{12 points}}$ Evaluate the following integrals:

i. $\displaystyle\int x\sec^2 x\, dx.$

$u = x \qquad dv = \sec^2 x$

$du = dx \qquad v = \tan x$ 0.5

$\displaystyle\int x \sec^2 x\, dx = \underbrace{x\tan x}_{0.5} - \underbrace{\int \tan x\, dx}_{0.5}$

$\displaystyle = x\tan x - \int \frac{\sin x}{\cos x}\, dx$ 0.5

$= \boxed{x\tan x + \ln|\cos x| + c}$ 1

ii. $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\, dx.$

$x = 3\tan u \Rightarrow dx = 3\sec^2 u\, du$

$\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\, dx = \int \frac{27\tan^3 u}{\sqrt{9\tan^2 u + 9}}\, 3\sec^2 u\, du$ 0.5

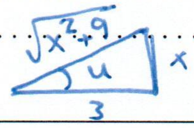
$\displaystyle = \int \frac{27\tan^3 u\, \cancel{3}\, \sec^{\cancel{2}\,1} u}{\cancel{3}\, \cancel{\sec u}}\, du$ 0.5 $\displaystyle = \int 27\tan^3 u \sec u\, du$ 0.5

$w = \sec u \Rightarrow dw = \sec u \tan u\, du$

$\displaystyle = \int 27(w^2 - 1)\, \cancel{\tan u}\, \cancel{\sec u}\, \frac{dw}{\cancel{\sec u}\, \cancel{\tan u}}$ 0.5

$\displaystyle = 27\left(\frac{w^3}{3} - w\right) + c = 27\left(\frac{\sec^3 u}{3} - \sec u\right) + c$ 0.5

$\displaystyle = 27\left(\frac{(\sqrt{x^2+9})^3}{27 * 3} - \frac{\sqrt{x^2+9}}{3}\right) + c$



$\displaystyle = \boxed{\frac{(\sqrt{x^2+9})^3}{3} - 9\sqrt{x^2+9} + c}$ 0.5

iii. $\displaystyle\int_0^2 \frac{e^x}{\sqrt{e^x-1}}\,dx.$

$$\lim_{R\to 0} \int_R^2 \frac{e^x}{\sqrt{e^x-1}}\,dx \quad 0.5$$

$$u = e^x - 1 \Rightarrow du = e^x\,dx \Rightarrow dx = \frac{du}{e^x}$$

$$= \lim_{R\to 0} \int_R^2 \frac{e^x}{\sqrt{u}}\,\frac{du}{e^x} \;0.5\; = \lim_{R\to 0} \int_R^2 u^{-1/2}\,du \quad 0.5$$

$$= \lim_{R\to 0} 2\sqrt{u}\,\Big|_R^2 \;0.5\; = \lim_{R\to 0} \left(2\sqrt{e^x-1}\right)\Big|_R^2 \quad 0.5$$

$$= \lim_{R\to 0}\left(2\sqrt{e^2-1} - 2\sqrt{e^R-1}\right) = \boxed{2\sqrt{e^2-1}} \quad 0.5$$

iv. $\displaystyle\int_0^1 \int_0^{\sqrt{x}} 2\,\sqrt{x}\,e^{x^2}\,dy\,dx.$

$$\int_0^1 \int_0^{\sqrt{x}} 2\sqrt{x}\,e^{x^2}\,dy\,dx = \int_0^1 \left(2\sqrt{x}\,e^{x^2} y\right)\Big|_0^{\sqrt{x}}\,dx \quad 1$$

$$= \int_0^1 2x\,e^{x^2}\,dx \;1\; = e^{x^2}\Big|_0^1 \quad 0.5$$

$$= e^1 - e^0$$

$$= \boxed{e-1} \quad 0.5$$

2. $\boxed{\text{7 points}}$

(a) $\boxed{\text{4 points}}$ Determine whether the following series converges or diverges:

i. $\displaystyle\sum_{k=3}^{\infty} \frac{(-3k)^k (k+1)^k}{k^{2k}}$.

by root test

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{(-3k)^k (k+1)^k}{k^{2k}}\right|} \quad 0.5$$

$$= \lim_{k\to\infty} \left|\frac{-3k(k+1)}{k^2}\right| \overset{0.5}{=} \lim_{k\to\infty} \left|\frac{-3k^2-3k}{k^2}\right|$$

$$= |-3| = 3 > 1 \quad 0.5$$

$\Rightarrow$ The series div. 0.5

ii. $\displaystyle\sum_{k=1}^{\infty} \frac{k^{-2}}{2+\sin^2 k}$.

by comparison test

$$\frac{k^{-2}}{2+\sin^2 k} > 0 \quad 0.5$$

$$\frac{k^{-2}}{2+\sin^2 k} < \frac{k^{-2}}{2} = \frac{1}{2k^2} \quad 0.5$$

since $\sum \frac{1}{2k^2}$ Conv. by P-series 0.5

$\Rightarrow \sum \frac{k^{-2}}{2+\sin^2 k}$ Conv. by comparison test 0.5

(b) $\boxed{\text{3 points}}$ Determine the interval and radius of convergence of the following power series:

$$\sum_{k=1}^{\infty} \frac{(x-4)^k}{\sqrt[3]{k}}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-4)^{k+1}}{\sqrt[3]{k+1}} \cdot \frac{\sqrt[3]{k}}{(x-4)^k}\right|$$

$$= \lim_{k\to\infty}\left|(x-4)\sqrt[3]{\frac{k}{k+1}}\right| = |x-4| < 1 \quad 0.5$$

$$-1 < x-4 < 1 \quad \Rightarrow \quad 3 < x < 5 \quad 0.5$$

$x=5$: $\sum \frac{(5-4)^k}{\sqrt[3]{k}} = \sum \frac{1}{\sqrt[3]{k}}$ div. by P-series 0.5

$x=3$: $\sum \frac{(3-4)^k}{\sqrt[3]{k}} = \sum \frac{(-1)^k}{\sqrt[3]{k}}$ conv. by alternating series test 0.5

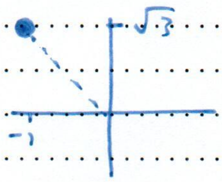
int. conv. $\boxed{3 \le x < 5}$ 0.5 radius conv. $\boxed{r = \frac{5-3}{2} = 1}$ 0.5

3. $\boxed{\text{6 points}}$

(a) $\boxed{\text{2 points}}$ Find all polar coordinate representation for the rectangular coordinate $(-1, \sqrt{3})$.

$$r^2 = x^2 + y^2 = (-1)^2 + (\sqrt{3})^2 = 4 \quad \Rightarrow \quad r = \pm 2 \quad 0.5$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{-1} = \tan^{-1} -\sqrt{3} = -\frac{\pi}{3} \quad 0.5$$



$$\left(2, \frac{-\pi}{3} + \pi + 2n\pi\right) = \boxed{\left(2, \frac{2\pi}{3} + 2n\pi\right)} \quad 0.5$$

$$\boxed{\left(-2, \frac{-\pi}{3} + 2n\pi\right)} \quad 0.5$$

(b) $\boxed{\text{2 points}}$ Find the slope of the tangent line to the polar curve $r = 3\sin\theta$ at $\theta = \frac{\pi}{4}$.

$$\text{slope} \quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{\frac{dy}{d\theta}(\frac{\pi}{4})}{\frac{dx}{d\theta}(\frac{\pi}{4})} \quad 0.5$$

$$= \frac{3\cos\theta\sin\theta + 3\sin\theta\cos\theta}{3\cos\theta\cdot\cos\theta - 3\sin\theta\cdot\sin\theta} \quad 0.5$$

$$= \frac{3\cos(\pi/4)\sin(\pi/4) + 3\sin(\pi/4)\cos(\pi/4)}{3\cos(\pi/4)\cos(\pi/4) - 3\sin(\pi/4)\sin(\pi/4)} \quad 0.5$$

$$= \frac{3(\frac{1}{2}) + 3(\frac{1}{2})}{3(\frac{1}{2}) - 3(\frac{1}{2})} \quad \text{undefine} \quad 0.5$$

(c) $\boxed{\text{2 points}}$ Show that the rectangular equation $x^2 - 3x + y^2 = 0$ is corresponding to the polar equation $r = 3\cos\theta$.

$$x^2 + y^2 - 3x = 0$$

$$r^2 - 3r\cos\theta = 0 \quad 1$$

$$r^2 = 3r\cos\theta \quad 0.5$$

$$\Rightarrow \quad r = 3\cos\theta \quad 0.5$$

4. $\boxed{\text{12 points}}$

(a) $\boxed{\text{3 points}}$ Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{3\cdot\sqrt{x^4+y^4}} = 0.$

$$\left|\frac{x^2y}{3\sqrt{x^4+y^4}} - 0\right| = \left|\frac{x^2y}{3\sqrt{x^4+y^4}}\right| < \left|\frac{x^2y}{3\sqrt{x^4}}\right| \quad \underline{\underline{1}}$$

0.S

$$= \left|\frac{x^2y}{3x^2}\right| = \left|\frac{y}{3}\right|$$

0.S

$$\lim_{(x,y)\to(0,0)} \left|\frac{y}{3}\right| = 0 \quad \text{0.S}$$

$$\Longrightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y}{3\cdot\sqrt{x^4+y^4}} = 0 \quad \text{0.S}$$

(b) $\boxed{\text{3 points}}$ Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4y}{(2x^2+y)^3}$ does not exist.

* $x=0 \quad y\to 0$

$$\underline{\underline{1}} \quad \lim_{(0,y)\to(0,0)} \frac{3(0)^4y}{(2x^2+y)^3} = 0$$

$=$

* $x\to 0 \quad y=0$

$$\underline{\underline{1}} \quad \lim_{(x,0)\to(0,0)} \frac{3x^4(0)}{(2x^2+y)^3} = 0$$

$\neq$

* $y=x^2$

$$\underline{\underline{1}} \quad \lim_{(x,x^2)\to(0,0)} \frac{3x^4x^2}{(2x^2+x^2)^3} = \lim_{(x,x^2)\to(0,0)} \frac{3x^6}{27x^6} = \frac{1}{9}$$

$\Longrightarrow$ DNE

(c) $\boxed{\text{3 points}}$ Let $f(x, y) = \sin(xy) + y^2 e^x$, compute f_{yx}.

$f_y(x,y) = x\cos(xy) + 2ye^x$

0.5 0.5

$f_{yx}(x,y) = (1)\cos(xy) - xy\sin(xy) + 2ye^x$

0.5 1 0.5

(d) $\boxed{\text{3 points}}$ Let $f(x, y) = \ln(x^2 + y^2)$, show that $f_{xx} + f_{yy} = 0$.

$f_x(x,y) = \dfrac{2x}{x^2+y^2}$ 0.5

$f_{xx}(x,y) = \dfrac{(2)(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2}$

$= \dfrac{2y^2 - 2x^2}{(x^2+y^2)^2}$ 0.5

$f_y(x,y) = \dfrac{2y}{x^2+y^2}$ 0.5

$f_{yy}(x,y) = \dfrac{(2)(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2}$

$= \dfrac{2x^2 - 2y^2}{(x^2+y^2)^2}$ 0.5

$f_{xx} + f_{yy} = \dfrac{2y^2 - 2x^2}{(x^2+y^2)^2} + \dfrac{2x^2 - 2y^2}{(x^2+y^2)^2} = 0$

1