Answer Sheet - Final (220 math)

Q(1)

$\boxed{1}$

P	Q	R	$\neg P$	$Q \Rightarrow R$	$\neg P \Rightarrow (Q \Rightarrow R)$	$P \vee R$	$Q \Rightarrow (P \vee R)$
T	T	T	F	T	$\boxed{T}$	T	$\boxed{T}$
T	T	F	F	F	$\boxed{T}$	T	$\boxed{T}$
T	F	T	F	T	$\boxed{T}$	T	$\boxed{T}$
T	F	F	F	T	$\boxed{T}$	T	$\boxed{T}$
F	T	T	T	T	$\boxed{T}$	T	$\boxed{T}$
F	T	F	T	F	$\boxed{F}$	F	$\boxed{F}$
F	F	T	T	T	$\boxed{T}$	T	$\boxed{T}$
F	F	F	T	T	$\boxed{T}$	F	$\boxed{T}$
					(*)		(**)

Then, $[\neg P \Rightarrow (Q \Rightarrow R)] \equiv [Q \Rightarrow (P \vee R)]$

Q(1)

$\boxed{2}$

We have to prove $\left[((P \vee Q) \wedge \neg P) \Rightarrow Q\right] \stackrel{?}{\equiv} t$

$$\text{L.H.S.} = \left[((P \vee Q) \wedge \neg P) \Rightarrow Q\right] \equiv \left[((P \wedge \neg P) \vee (Q \wedge \neg P)) \Rightarrow Q\right]$$

$$\equiv \left[(f \vee (Q \wedge (\neg P))) \Rightarrow Q\right]$$

$$\equiv \left[(Q \wedge \neg P) \Rightarrow Q\right]$$

$$\equiv \left[\neg(Q \wedge \neg P) \vee Q\right]$$

$$\equiv (\neg Q \vee P) \vee Q$$

$$\equiv (P \vee \neg Q) \vee Q$$

$$\equiv P \vee (\neg Q \vee Q)$$

$$\equiv P \vee t$$

$$\equiv t = \text{R.H.S.}$$

Then, $((P \vee Q) \wedge \neg P) \Rightarrow Q$ is a tautology.

Q(1)

$\boxed{3}$

$\{3\}$

(a) $(A \cap C)' = \{2, 3, 5, 6, 7, 8, 9, 10\}$.

(b) $(A \cup B)' = \{5, 7, 10\}$

(c) $A - C = \{3, 8\}$.

Q(2)

$\boxed{1}$

Let $P(n): \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n\times(n+1)} = \frac{n}{n+1}$, $n \geq 1$.

① Basis step:

$$\left.\begin{array}{l} \text{L.H.S. of } P(1) = \frac{1}{1\times 2} = \frac{1}{2}. \\ \text{R.H.S. of } P(1) = \frac{1}{1+1} = \frac{1}{2} \end{array}\right] \text{L.H.S. of } P(1) = \text{R.H.S. of } P(1).$$

Then, $P(1)$ is true.

② Inductive step.

(i) Inductive hypothesis

Suppose that $P(K)$ is true, for some $k \geq 1$.

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k\times(k+1)} = \frac{k}{k+1} \longrightarrow (*).$$

(ii) We have to prove $P(K+1)$ is true.

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k\times(k+1)} + \frac{1}{(k+1)\times(k+2)} \stackrel{?}{=} \frac{k+1}{k+2}$$

$$\text{L.H.S.} = \frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad (\text{from } (*))$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \text{R.H.S.}$$

Then, $P(k+1)$ is true.

So, by Mathematical induction, $P(n)$ is true, $\forall n \geq 1$.

Q (2)

2

Let P: $7m-4$ is odd.

Q: $5m+3$ is even.

A $P \Rightarrow Q$ (By Direct Proof)

1 Assume that $7m-4$ is odd.

$$\boxed{7m-4=2k+1,\ k\in\mathbb{Z}}$$

2 We have to prove $5m+3$ is even

$$\boxed{5m+3\overset{?}{=}2t,\ t\overset{?}{\in}\mathbb{Z}}$$

3
$$\begin{aligned} 5m+3 &= 7m-2m+3-4+4 \\ &= 7m-4-2m+7 \\ &= 2k+1-2m+7 \\ &= 2k-2m+8 \\ &= 2(k-m+4) \\ &= 2t, \text{ where } t=k-m+4\in\mathbb{Z} \end{aligned}$$

4 Therefore, $5m+3$ is even.

5 Hence, $P\Rightarrow Q$.

B $Q \Rightarrow P$ (By Direct Proof)

1 Assume that $5m+3$ is even.

$$\boxed{5m+3=2p,\ p\in\mathbb{Z}}$$

2 We have to prove $7m-4$ is odd.

$$\boxed{7m-4\overset{?}{=}2s+1,\ s\overset{?}{\in}\mathbb{Z}}$$

3
$$\begin{aligned} 7m-4 &= 5m+2m-4+3-3 \\ &= 5m+3+2m-7 \\ &= 2p+2m-7 \\ &= 2p+2m-8+1 \\ &= 2(p+m-4)+1 \\ &= 2s+1, \text{ where } s=p+m-4\in\mathbb{Z} \end{aligned}$$

4 Therefore, $7m-4$ is odd.

5 Hence, $Q\Rightarrow P$.

From **A** and **B**, we get $P \Leftrightarrow Q$.

Q (2)

$\boxed{3}$

L.H.S. $= (A \cup B) - C$

$= (A \cup B) \cap C'$

$= (A \cap C') \cup (B \cap C')$

$= (A - C) \cup (B - C)$ = R.H.S.

Then, $(A \cup B) - C = (A - C) \cup (B - C)$.

Q (3)

$\boxed{1}$

let (x,y) be an arbitrary element.

$(x,y) \in A \times (B \cup C)$

$\Leftrightarrow (x \in A) \wedge (y \in B \cup C)$ (Def. of $\times$)

$\Leftrightarrow (x \in A) \wedge ((y \in B) \vee (y \in C))$ (Def. of $\cup$)

$\Leftrightarrow [(x \in A) \wedge (y \in B)] \vee [(x \in A) \wedge (y \in C)]$ (Distributive Law)

$\Leftrightarrow [(x,y) \in A \times B] \vee [(x,y) \in A \times C]$ (Def. of $\times$)

$\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C)$. (Def. of $\cup$)

Then, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Q(3)

$\boxed{2}$

ⓐ $R = \{(1,2), (1,4), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)\}$

$S = \{(1,1), (1,2), (1,4), (2,2), (2,4), (4,4)\}$.

ⓑ $\text{Dom}(R) = \{1,2,4\} = A$

$\text{Rng}(S) = \{1,2,4\} = A$.

ⓒ $S \circ R = \{(1,2), (1,4), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)\}$

ⓓ $R^{-1} \circ S^{-1} = (S \circ R)^{-1} = \{(2,1), (4,1), (1,2), (2,2), (4,2), (1,4), (2,4), (4,4)\}$

ⓔ S is an antisymmetric
since $(\forall x, y \in A)[(x,y) \in S \wedge (y,x) \in S \Rightarrow x = y]$.

Q (4)

[1] $(x,y)\in R \Leftrightarrow x+y=2k,\ k\in\mathbb{Z}$.

(i) Reflexive: $(\forall x\in\mathbb{Z})((x,x)\in R)$.

Let $x\in\mathbb{Z}$.

$x+x=2x,\ x\in\mathbb{Z} \Rightarrow (x,x)\in R$.

Then, R is reflexive on $\mathbb{Z}$.

(ii) Symmetric. $(\forall x,y\in\mathbb{Z})((x,y)\in R \Rightarrow (y,x)\in R)$.

Let $x,y\in\mathbb{Z}$.

$$\begin{aligned}(x,y)\in R &\Rightarrow x+y=2k,\ k\in\mathbb{Z}\\ &\Rightarrow y+x=2k,\ k\in\mathbb{Z}\\ &\Rightarrow (y,x)\in R.\end{aligned}$$

Then, R is symmetric on $\mathbb{Z}$.

(iii) Transitive: $(\forall x,y,z\in\mathbb{Z})[(x,y)\in R \wedge (y,z)\in R \Rightarrow (x,z)\in R]$

Let $x,y,z\in\mathbb{Z}$.

$$\begin{aligned}((x,y)\in R)\wedge((y,z)\in R) &\Rightarrow (x+y=2p,\ p\in\mathbb{Z})\wedge(y+z=2m,\ m\in\mathbb{Z})\\ &\Rightarrow x+z=2p+2m-2y\\ &\Rightarrow x+z=2(p+m-y)\\ &\Rightarrow x+z=2t,\ \text{where } t=p+m-y\in\mathbb{Z}\\ &\Rightarrow (x,z)\in R.\end{aligned}$$

Then, R is transitive on $\mathbb{Z}$.

From (i), (ii) and (iii), we have R is an equivalence relation on $\mathbb{Z}$.

Equivalence classes:

$[0]=\{y\in\mathbb{Z} \mid (y,0)\in R\}=\{y\in\mathbb{Z} \mid y=2k,\ k\in\mathbb{Z}\}=\{\dots,-4,-2,0,2,4,\dots\}=E$

$[1]=\{y\in\mathbb{Z} \mid (y,1)\in R\}=\{y\in\mathbb{Z} \mid y=2k-1,\ k\in\mathbb{Z}\}=\{\dots,-3,-1,1,3,\dots\}=O$

Q (4)

$\boxed{2}$

(i) 1-1 $(\forall x_1, x_2 \in \mathbb{R}-\{1\})\ [f(x_1)=f(x_2) \Rightarrow x_1=x_2]$

let $x_1, x_2 \in \mathbb{R}-\{1\}$.

$$f(x_1)=f(x_2) \Rightarrow \frac{2x_1-1}{x_1-1} = \frac{2x_2-1}{x_2-1}$$

$$\Rightarrow \cancel{2x_1x_2} - x_2 - 2x_1 + \cancel{1} = \cancel{2x_1x_2} - 2x_2 - x_1 + \cancel{1}$$

$$\Rightarrow -2x_1 + x_1 = -2x_2 + x_2$$

$$\Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2.$$

Then, f is 1-1.

(ii) Onto $(\forall y \in \mathbb{R}-\{2\})\ (\exists x \in \mathbb{R}-\{1\})$ s.t $(f(x)=y)$.

let $y \in \mathbb{R}-\{2\}$ $(y \neq 2)$

$\exists x = \frac{1-y}{2-y} \in \mathbb{R}-\{1\}$ s.t

$$f(x) = f\left(\frac{1-y}{2-y}\right) = \frac{2\left(\frac{1-y}{2-y}\right)-1}{\frac{1-y}{2-y}-1}$$

$$= \frac{2(1-y)-(2-y)}{1-y-(2-y)}$$

$$= \frac{-y}{-1} = y.$$

Then, f is onto.

$f(x)=y$

$$\frac{2x-1}{x-1} = y \Rightarrow 2x-1 = xy-y$$

$$\Rightarrow 2x - xy = 1-y$$

$$\Rightarrow x(2-y) = 1-y$$

$$\Rightarrow x = \frac{1-y}{2-y},\ y \neq 2$$

if $x=1 \Rightarrow \frac{1-y}{2-y} = 1 \Rightarrow 1-y = 2-y$

$\Rightarrow 1 = 2$ ※

Then, $x \neq 1$

$x = \frac{1-y}{2-y} \in \mathbb{R}\setminus\{1\}$.

From (i) and (ii), f is one-to-one correspondence.

$$f^{-1}(y) = \frac{1-y}{2-y}$$

Q(4)

3

We have to prove $g \circ f$ is one-to-one.

$$(\forall x_1, x_2 \in A)\left[(g \circ f)(x_1) = (g \circ f)(x_2) \overset{?}{\Longrightarrow} x_1 = x_2\right]$$

Let $x_1, x_2 \in A$ and $(g \circ f)(x_1) = (g \circ f)(x_2)$.

$(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ [since $f(x_1), f(x_2) \in B$ and g is 1-1]

$\Rightarrow x_1 = x_2$ [since $x_1, x_2 \in A$ and f is 1-1]

Then, $g \circ f$ is 1-1.