Chapter 4 : Inductance

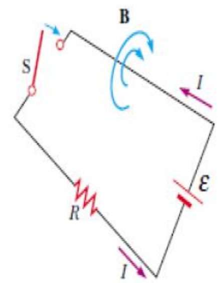
Chapter 4 Outline:

Chapter 32 from the text book
4.1 Self-inductance
4.2 RLCircuits
4.3 Energy in a magnetic field
4.4 Mutual inductance
4.5 Oscillation in an LC circuit
4.6 RLC circuit.

4.1 Self-inductance

Consider a circuit consisting of a switch, a resistor, and a source of emf, as shown in Figure. When the switch is thrown to its closed position, the current does not immediately jump from zero to its maximum value $\mathcal{E}/R$.

The direction of the induced emf is such that it would cause an induced current in the loop, which would establish a magnetic field opposing the change in the original magnetic field. Thus, the direction of the induced emf is opposite the direction of the emf of the source; this results in a gradual rather than instantaneous increase in the current to its final equilibrium value.



Because of the direction of the induced emf, it is also called a back emf. This effect is called **self-induction** because the changing flux through the circuit and the resultant induced emf arise from the circuit itself. The emf $\mathcal{E}_L$ set up in this case is called **a self-induced emf**.

4.1 Self-inductance

Another example of self-induction, as shown in Figure, consider a coil wound on a cylindrical core. Assume that the current in the coil either increases or decreases with time.

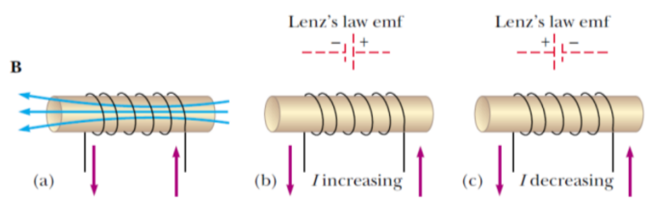


Figure 1

(a) A current in the coil produces a magnetic field directed to the left.
(b) If the current increases, the increasing magnetic flux creates an induced emf in the coil having the polarity shown by the dashed battery.
(c) The polarity of the induced emf reverses if the current decreases.

4.1 Self-inductance

we recall from Faraday's law that the induced emf is equal to the negative of the time rate of change of the magnetic flux. The magnetic flux is proportional to the magnetic field due to the current, which in turn is proportional to the current in the circuit. Therefore, a self-induced emf is always proportional to the time rate of change of the current. For any coil, we find that

$$\mathcal{E}_L = -L\frac{dI}{dt}$$

where L is a proportionality constant—called the **inductance** of the coil—that depends on the geometry of the coil and other physical characteristics.

Inductance of an N-turn coil: $$L = \frac{N\Phi_B}{I}$$

where it is assumed that the same magnetic flux passes through each turn.
We can also write the inductance as the ratio.

$$L = -\frac{\mathcal{E}_L}{dI/dt}$$

4.1 Self-inductance

The SI unit of inductance is the henry (H), which is 1 volt-second per ampere:

$$1\text{ H} = 1\frac{\text{V}\cdot\text{s}}{\text{A}}$$

Example 32.1 Inductance of a Solenoid

Find the inductance of a uniformly wound solenoid having N turns and length ℓ. Assume that ℓ is much longer than the radius of the windings and that the core of the solenoid is air.

4.1 Self-inductance

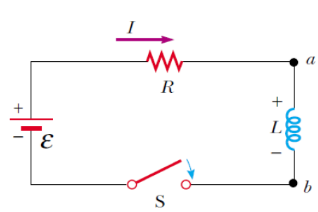
Example 32.2 Calculating Inductance and emf

(A) Calculate the inductance of an air-core solenoid containing 300 turns if the length of the solenoid is 25.0 cm and its cross-sectional area is 4.00 cm^2.

(B) Calculate the self-induced emf in the solenoid if the current it carries is decreasing at the rate of 50.0 A/s.

4.2 RL Circuits

Consider the circuit shown in Figure, which contains a battery of negligible internal resistance. This is an RL circuit because the elements connected to the battery are a resistor and an inductor. Suppose that the switch S is open for $t < 0$ and then closed at $t = 0$. The current in the circuit begins to increase, and a back emf that opposes the increasing current is induced in the inductor.
Because the current is increasing, dI/dt is positive; thus, $\mathcal{E}_L$ is negative.



4.2 RL Circuits

We can apply Kirchhoff's loop rule to this circuit, traversing the circuit in the clockwise direction:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

To find solution for this equation, we change variables for convenience, letting $x = (\mathcal{E}/R) - I$, so that $dx = -dI$. With these substitutions, we can write the equation as

$$x + \frac{L}{R}\frac{dx}{dt} = 0 \quad \Longrightarrow \quad \frac{dx}{x} = -\frac{R}{L}\,dt$$

Integrating this last expression, we have

$$\int_{x_0}^{x}\frac{dx}{x} = -\frac{R}{L}\int_0^t dt \quad \Longrightarrow \quad \ln\frac{x}{x_0} = -\frac{R}{L}t$$

where x_0 is the value of x at time t = 0. Taking the antilogarithm of this result, we obtain

$$x = x_0 e^{-Rt/L}$$

4.2 RL Circuits

Because I = 0 at t = 0, we note from the definition of x that $x_0 = \mathcal{E}/R$. Hence, this last expression is equivalent to

$$\frac{\mathcal{E}}{R} - I = \frac{\mathcal{E}}{R}e^{-Rt/L} \quad \Longrightarrow \quad I = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$

This expression shows how the inductor effects the current. The current does not increase instantly to its final equilibrium value when the switch is closed but instead increases according to an exponential function.
We can also write this expression as

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})$$

Where $\tau = \frac{L}{R}$ is the time constant of the RL circuit.

Physically, τ is the time interval required for the current in the circuit to reach $(1 - e^{-1}) = 0.632 = 63.2\%$ of its final value $\mathcal{E}/R$.

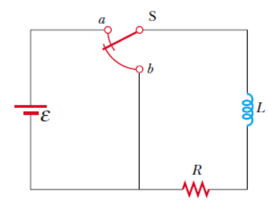
4.2 RL Circuits

Consider the RL circuit shown in Figure. The curved lines on the switch S represent a switch that is connected either to a or b at all times.
Applying Kirchhoff's loop rule to the right-hand loop at the instant the switch is thrown from a to b, we obtain:

$$IR + L\frac{dI}{dt} = 0$$

the solution of this differential equation is

$$I = \frac{\mathcal{E}}{R}e^{-t/\tau} = I_0 e^{-t/\tau}$$



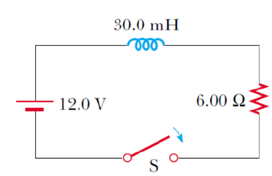
4.2 RL Circuits

Example 32.3 Time Constant of an *RL* Circuit

(A) Find the time constant of the circuit shown in Figure 32.11a.

(B) The switch in Figure 32.11a is closed at $t = 0$. Calculate the current in the circuit at $t = 2.00$ ms.

(C) Compare the potential difference across the resistor with that across the inductor.



4.3 Energy in a Magnetic Field:

The total energy stored in the inductor is :

$$U = \tfrac{1}{2} L I^2$$

We can also determine the energy density of a magnetic field. Consider a solenoid whose inductance is given by: $L = \mu_0 n^2 A\ell$

The magnetic field of a solenoid is given by:

$$B = \mu_0 n I$$

Substituting the expression for L and $I = B/\mu_0 n$ into the Energy Equation

$$U = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \mu_0 n^2 A\ell \left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0} A\ell$$

Because Al is the volume of the solenoid, the magnetic energy density, or the energy stored per unit volume in the magnetic field of the inductor is:

$$u_B = \frac{U}{A\ell} = \frac{B^2}{2\mu_0}$$

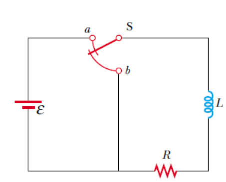
4.3 Energy in a Magnetic Field:

Example 1 :

A 24V battery is connected in series with a resistor and an inductor, where R = 8.0Ω and L = 4.0H. Find the energy stored in the inductor when the current reaches its maximum value?

Example 2 :

Consider the RL circuit shown in Figure. Recall that the current in the right-hand loop decays exponentially with time according to the expression , where $I_0 = \mathcal{E}/R$ is the initial current in the circuit and T = L/R is the time constant. Show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.

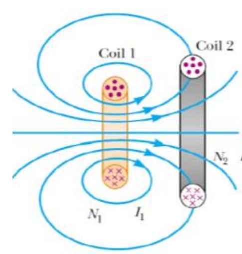


4.4 Mutual Inductance

Consider the two closely wound coils of wire shown in Figure. The current I_1 in coil ***1***, which has N_1 turns, creates a magnetic field. Some of the magnetic field lines pass through coil ***2***, which has N_2 turns. The magnetic flux caused by the current in coil 1 and passing through coil ***2*** is represented by ϕ_{12}. In analogy to equation, we define the mutual inductance M_{12} of coil ***2*** with respect to coil ***1*** as:

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1}$$

If the current I_1 varies with time, we see from Faraday's law and the last equation that the emf induced by coil 1 in coil 2 is:



$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{12}}{dt} = -N_2 \frac{d}{dt}\left(\frac{M_{12} I_1}{N_2}\right) = -M_{12} \frac{dI_1}{dt}$$

If the current I2 varies with time, the emf induced by coil 2 in coil 1 is:

$$\mathcal{E}_1 = -M_{21} \frac{dI_2}{dt}$$

4.4 Mutual Inductance

In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing.

It can be shown that M12 = M21 = M, thus,

$$\mathcal{E}_2 = -M \frac{dI_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M \frac{dI_2}{dt}$$

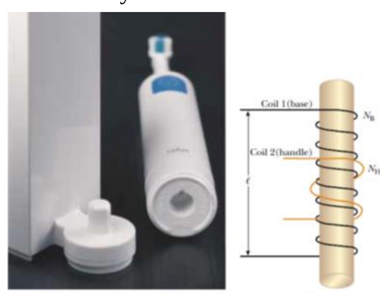
These two equations are similar in form to Equation of the self-induced emf

$\mathcal{E} = -L(dI/dt)$.

The unit of mutual inductance is the henry.

4.4 Mutual Inductance

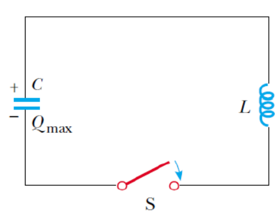
Example: **"Wireless" Battery Charger** An electric toothbrush has a base designed to hold the toothbrush handle when not in use. As shown in Figure a, the handle has a cylindrical hole that fits loosely over a matching cylinder on the base. When the handle is placed on the base, a changing current in a solenoid inside the base cylinder induces a current in a coil inside the handle. This induced current charges the battery in the handle. We can model the base as a solenoid of length ℓ with N_B turns (Fig. b), carrying a current I, and having a cross sectional area A. The handle coil contains N_H turns and completely surrounds the base coil. Find the mutual inductance of the system.



(a) (b)

4.5 Oscillations in an LC Circuit:

When a capacitor is connected to an inductor as illustrated in Figure, the combination is an LC circuit.



Let us consider some arbitrary time t after the switch is closed, so that the capacitor has a charge $Q < Q_{max}$ and the current is $I < I_{max}$. At this time, both circuit elements store energy, but the sum of the two energies must equal the total initial energy U stored in the fully charged capacitor at t = 0:

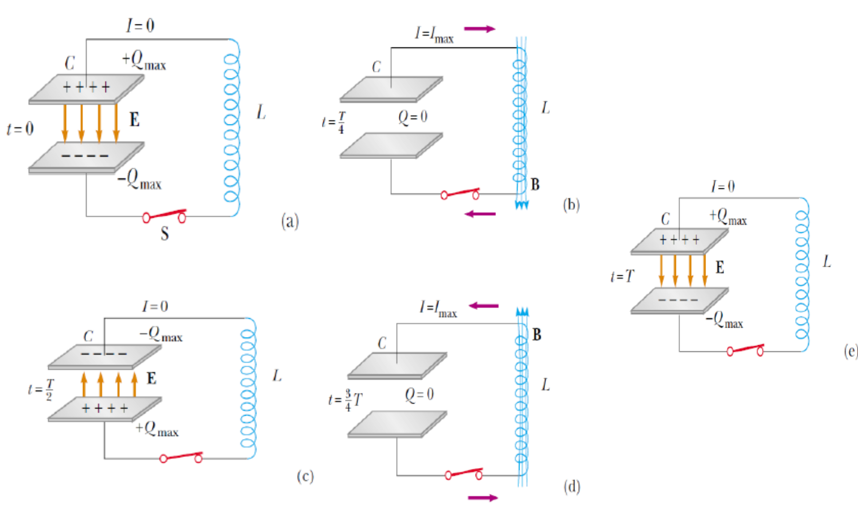
$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2$$

4.5 Oscillations in an LC Circuit:

Because we have assumed the circuit resistance to be zero and we ignore electromagnetic radiation, no energy is transformed to internal energy and none is transferred out of the system of the circuit. Therefore, the total energy of the system must remain constant in time.
This means that dU/dt = 0. Therefore, by differentiating the last equation with respect to time while noting that Q and I vary with time, we obtain:

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

4.5 Oscillations in an LC Circuit:



4.5 Oscillations in an LC Circuit:

Charge as a function of time for an ideal LC circuit:

$$Q = Q_{\max}\cos(\omega t + \phi)$$

where Q_{max} is the maximum charge of the capacitor and the angular frequency ω is

$$\omega = \frac{1}{\sqrt{LC}}$$

Current as a function of time for an ideal LC current:

$$I = \frac{dQ}{dt} = -\omega Q_{\max}\sin(\omega t + \phi)$$

To determine the value of the phase angle Φ, we examine the initial conditions, which in our situation require that at t = 0, I=0 and Q =Q_{max}. Setting I=0 at t=0 in the last equation, we have

$$0 = -\omega Q_{\max}\sin\phi \Longrightarrow \phi = 0$$

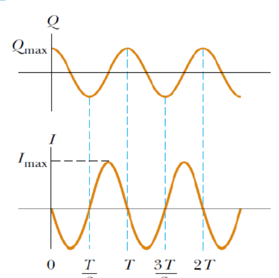
which shows that Φ = 0.

4.5 Oscillations in an LC Circuit:

Therefore, in our case, the expressions for Q and I are

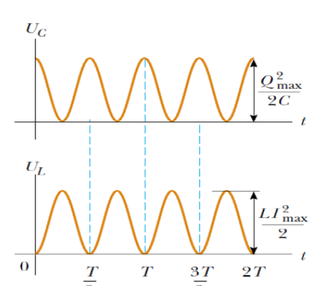
$$Q = Q_{\max}\cos\omega t$$

$$I = -\omega Q_{\max}\sin\omega t = -I_{\max}\sin\omega t$$



Let us return to the energy discussion of the LC circuit. we find that the total energy is

$$U = U_C + U_L = \frac{Q_{\max}^2}{2C}\cos^2\omega t + \frac{1}{2}LI_{\max}^2\sin^2\omega t$$



4.5 Oscillations in an LC Circuit:

Example 32.7 Oscillations in an *LC* Circuit

In Figure 32.20, the capacitor is initially charged when switch S_1 is open and S_2 is closed. Switch S_2 is then opened, removing the battery from the circuit, and the capacitor remains charged. Switch S_1 is then closed, so that the capacitor is connected directly across the inductor.

(A) Find the frequency of oscillation of the circuit.

(B) What are the maximum values of charge on the capacitor and current in the circuit?

(C) Determine the charge and current as functions of time.

4.6 The RLC Circuit:

The RLC circuit consisting of a resistor, an inductor, and a capacitor connected in series.

The total energy stored in the RLC circuit at any time is given by

$$LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2R$$

This total energy is no longer constant, as it was in the LC circuit, because the resistor causes transformation to internal energy. Because the rate of energy transformation to internal energy within a resistor is I^2R , we have

$$\frac{dU}{dt} = -I^2R$$

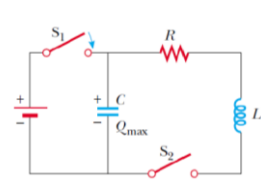
The total energy equation can be written as:

$$LI\frac{d^2Q}{dt^2} + I^2R + \frac{Q}{C}I = 0$$

Now we divide through by I :

$$L\frac{d^2Q}{dt^2} + IR + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$



4.6 The RLC Circuit:

The RLC circuit is analogous to the damped harmonic oscillator. When R is small, a situation analogous to light damping in the mechanical oscillator, the solution of Kirchhoff is

$$Q = Q_{\max} e^{-Rt/2L} \cos \omega_d t$$

where ω_d, the angular frequency at which the circuit oscillates, is given by

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L} \right)^2 \right]^{1/2}$$