Chapter 3

Vectors

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Outline

3.1 • Coordinate Systems

3.2 • Vector and Scalar Quantities

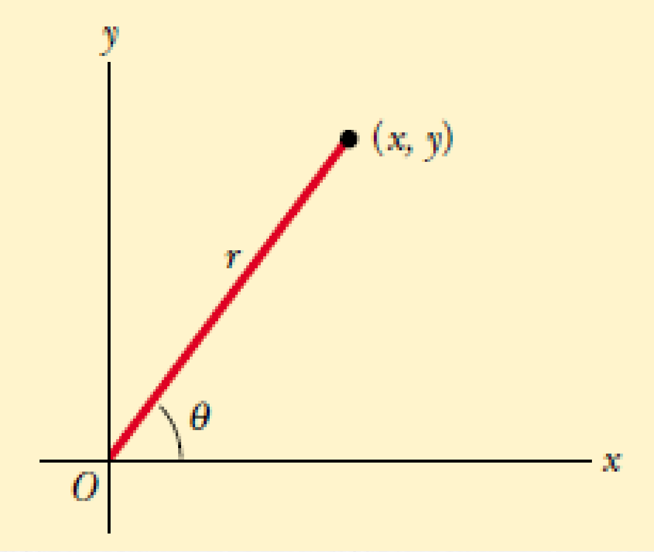
3.3 • Some Properties of Vectors

3.4 • Components of a Vector and Unit Vectors

3.1 Coordinate Systems

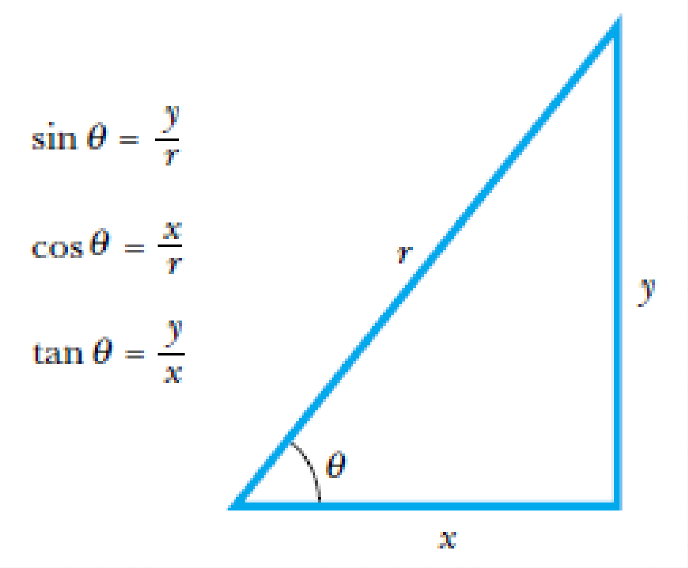
Cartesian coordinate system

(x, y)



polar coordinate system

(r, θ)

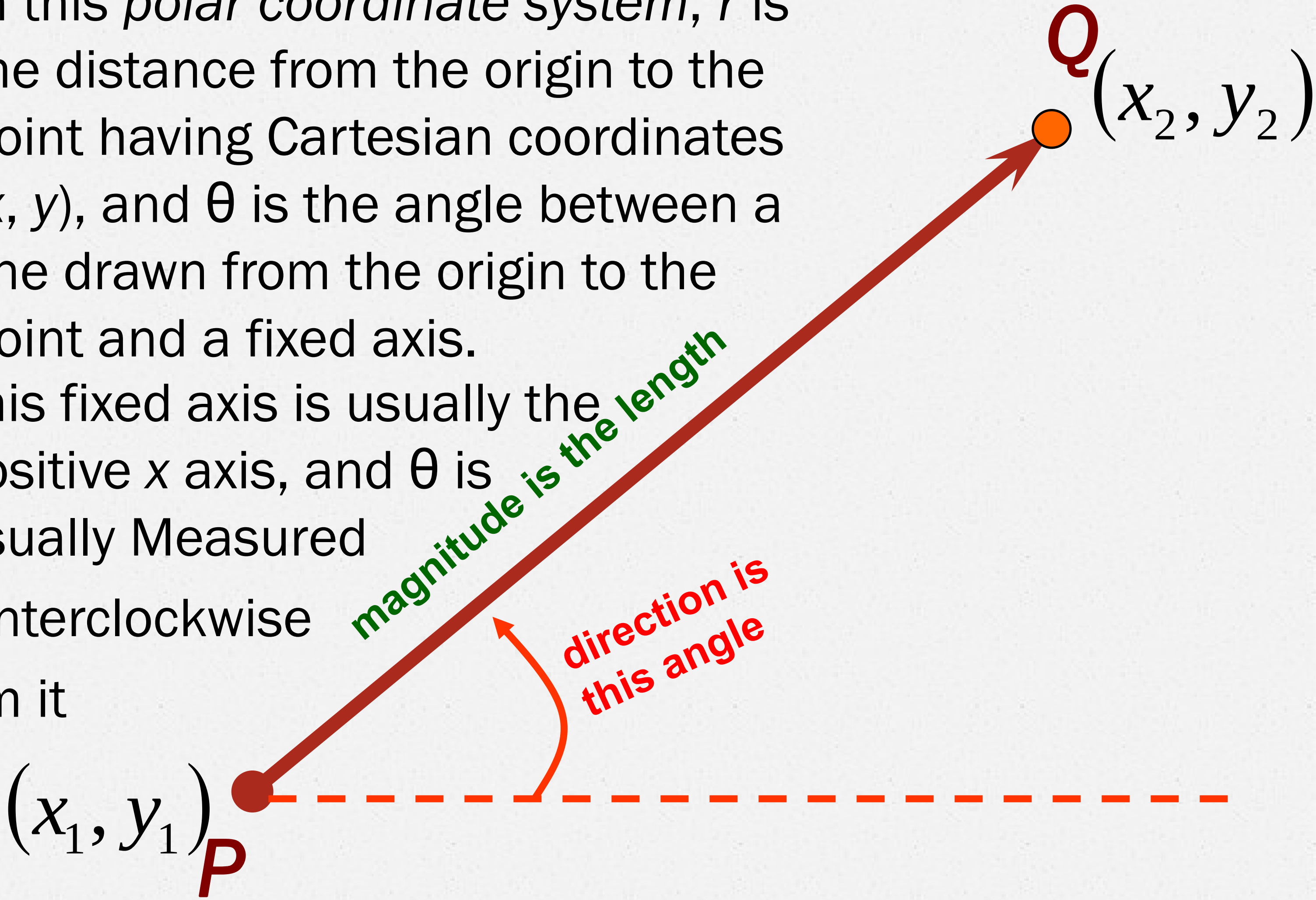


polar coordinate system(r, θ)

- In this *polar coordinate system*, *r* is the distance from the origin to the point having Cartesian coordinates (*x*, *y*), and θ is the angle between a line drawn from the origin to the point and a fixed axis.
- This fixed axis is usually the positive *x* axis, and θ is usually Measured

Counterclockwise

from it



Physics Mathematics

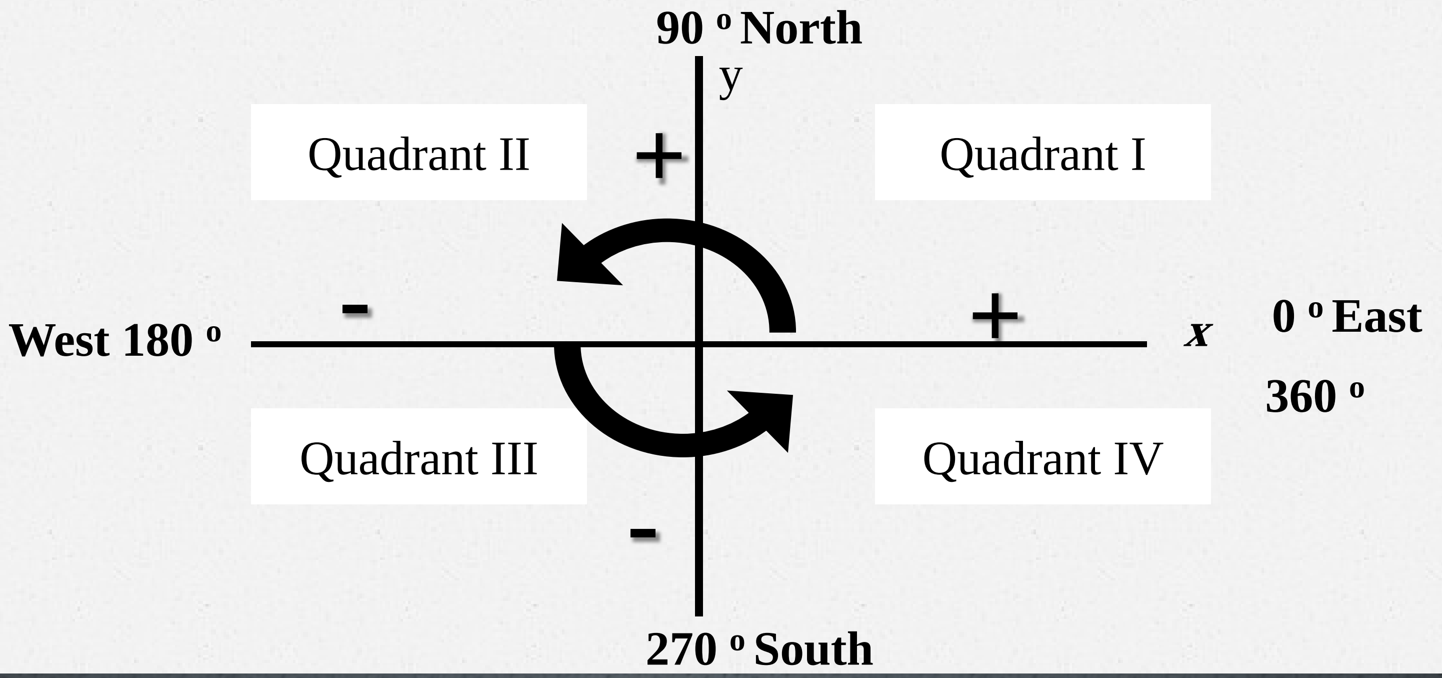
Up = +

Down = -

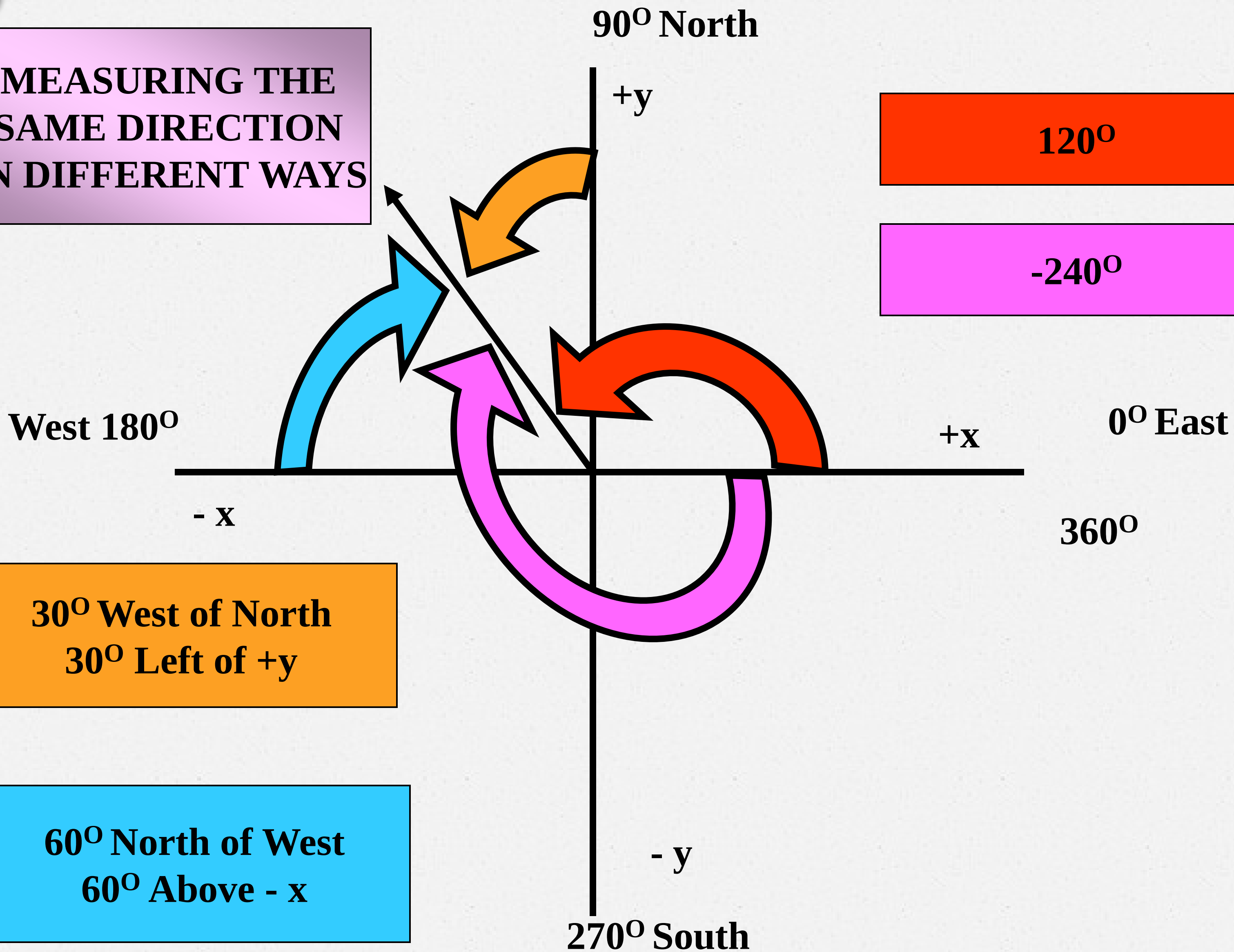
Right = +

Left = -

Rectangular Coordinates



MEASURING THE SAME DIRECTION IN DIFFERENT WAYS



120^{O}

-240^{O}

30^{O} West of North
30^{O} Left of +y

60^{O} North of West
60^{O} Above - x

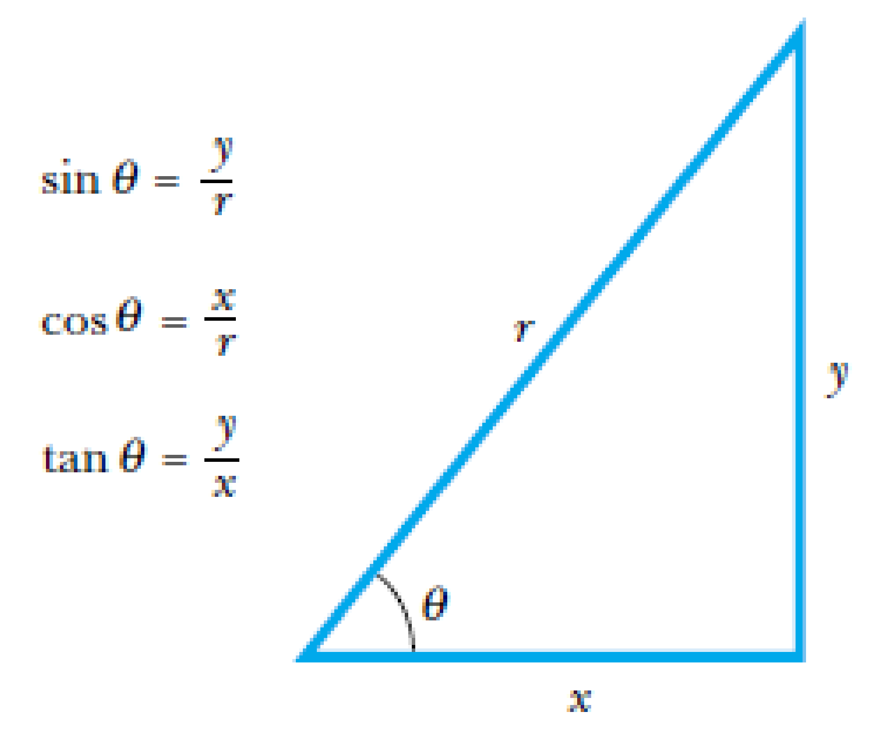
Trigonometry Review

- Therefore, starting with the plane polar coordinates of any point, we can obtain the Cartesian coordinates

$$y = R \sin \theta$$

$$x = R \cos \theta$$

$$R^2 = x^2 + y^2$$



R is always positive; θ is from + x axis

Example 3.1 Polar Coordinates

The Cartesian coordinates of a point in the xy plane are $(x, y) = (-3.50, -2.50)$ m, as shown in Figure 3.3. Find the polar coordinates of this point.

Solution For the examples in this and the next two chapters we will illustrate the use of the General Problem-Solving Strategy outlined at the end of Chapter 2. In subsequent chapters, we will make fewer explicit references to this strategy, as you will have become familiar with it and should be applying it on your own. The drawing in Figure 3.3 helps us to *conceptualize* the problem. We can *categorize* this as a plug-in problem. From Equation 3.4,

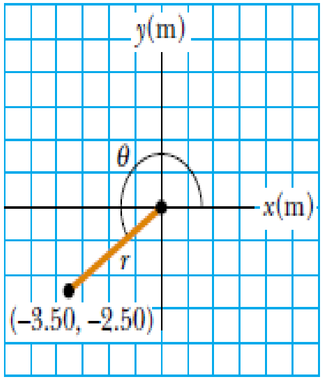
$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} = 4.30\text{ m}$$

and from Equation 3.3,

$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = 216°$$

Note that you must use the signs of x and y to find that the point lies in the third quadrant of the coordinate system. That is, $\theta = 216°$ and not $35.5°$.



Active Figure 3.3 (Example 3.1) Finding polar coordinates when Cartesian coordinates are given.

At the Active Figures link at http://www.pse6.com, you can move the point in the xy plane and see how its Cartesian and polar coordinates change.

3.2 Vector and Scalar Quantities

- Some physical quantities are scalar quantities whereas others are vector quantities.

Scalar

- A **scalar** quantity is completely specified by a single value with an appropriate unit and has no direction.
- Temperature.
- Time.
- Volume.
- Mass

Vector

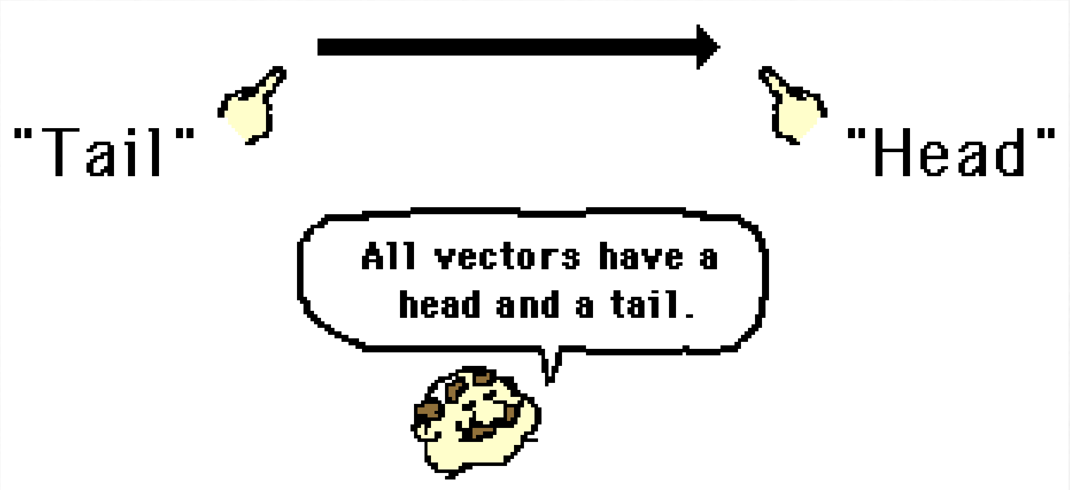
- A **vector quantity** is completely specified by a number and appropriate units plus a direction.
- Velocity.
- Displacement.
- Force.
- Electric Field

Vectors

- To represent a vector quantity **A**:

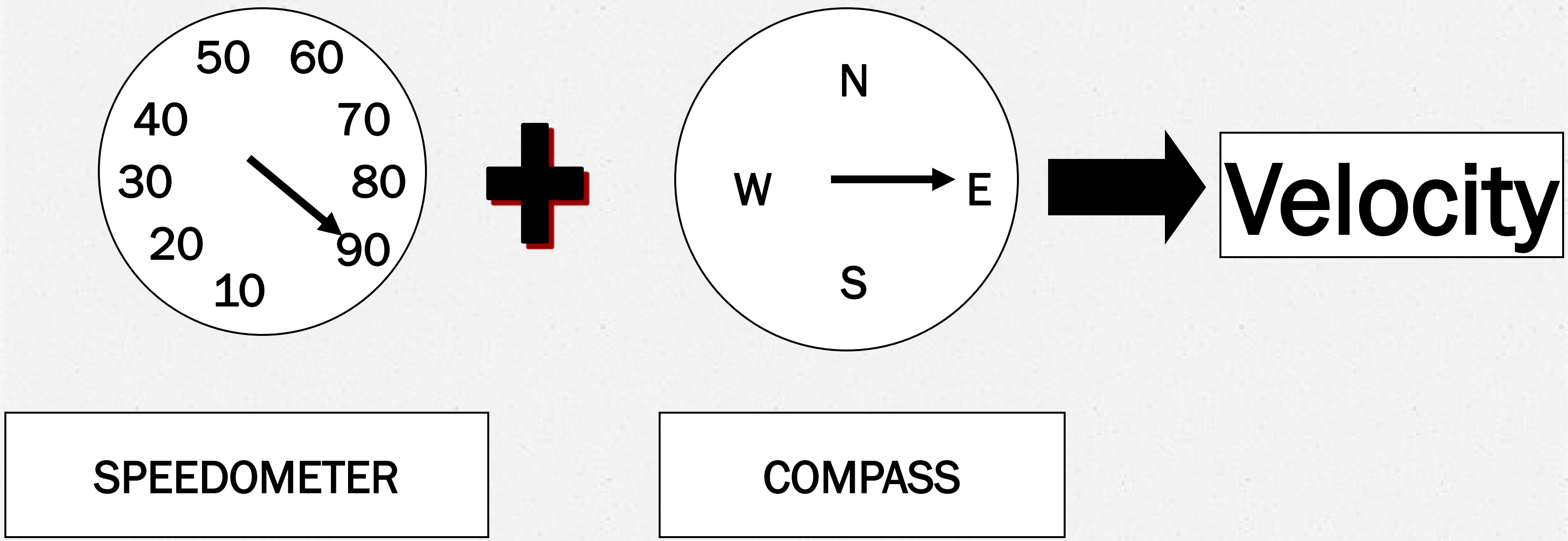
an arrow is written over the symbol for the vector $\vec{A}$

- The magnitude of the vector **A** is written either ***A*** or $|A|$
- The magnitude of a vector has physical units, such as meters for displacement or meters per second for velocity. The magnitude of a vector is *always* a positive number.



Speed & Velocity

- Speed and velocity are **not** the **same**.
- **Velocity** requires a **direction** therefore a **vector** quantity.
- **Speed** tells us how fast we are going but **not** which **way**.
- Speed is a **scalar** (direction doesn't count!)



3.3 Some Properties of Vectors

- **Equality of Two Vectors:**

For many purposes, two vectors A and B may be defined to be equal if they have the same magnitude and point in the same direction.

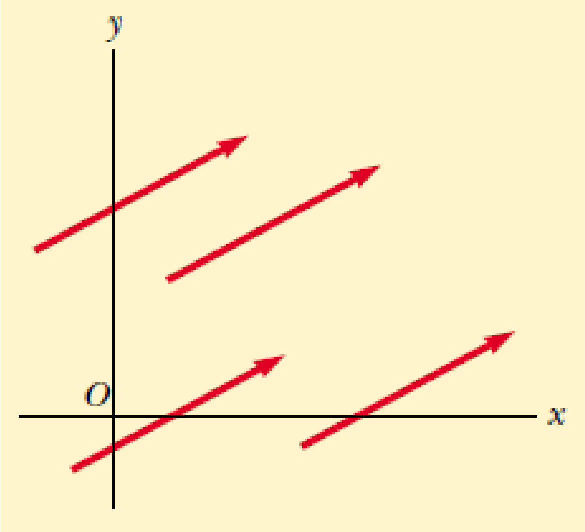


Figure 3.5 These four vectors are equal because they have equal lengths and point in the same direction.

Blue and orange vectors have same magnitude but different direction.

Blue and purple vectors have same magnitude and direction so they are equal.

Blue and green vectors have same direction but different magnitude.

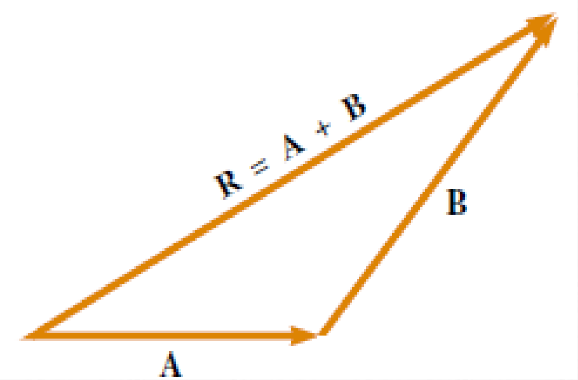
Two vectors are equal if they have the same direction and magnitude (length).

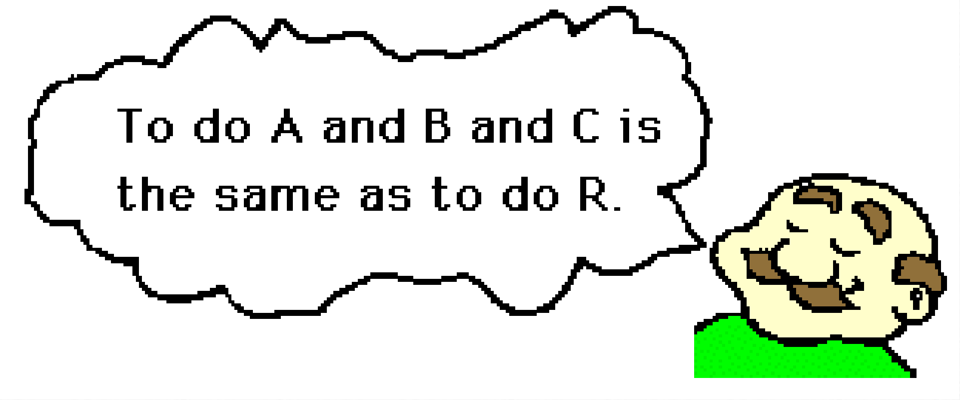
Adding Vectors:

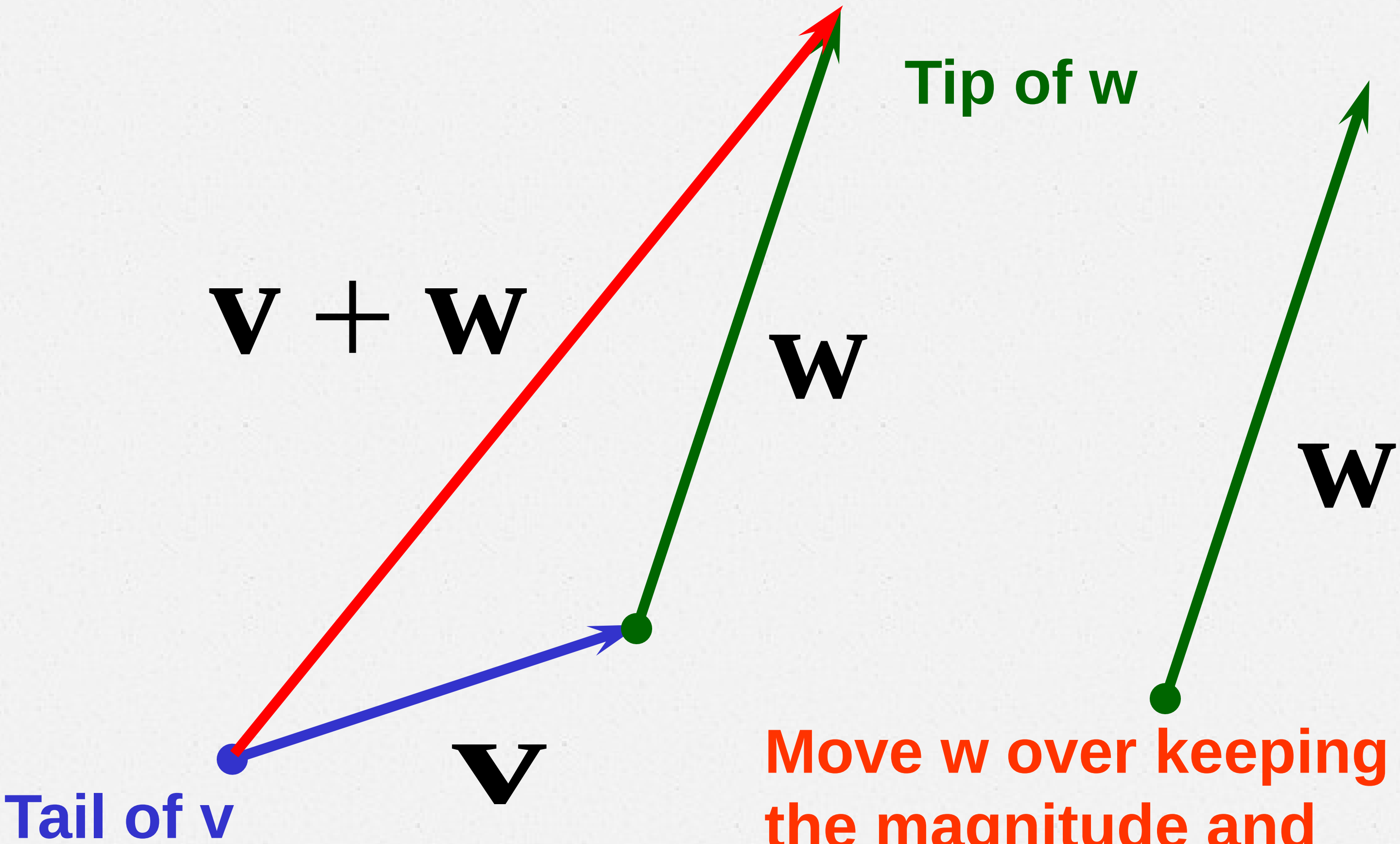
Head to Tail Method or (tip to tail)

To add vector B to vector A, first draw vector A on graph paper, with its magnitude represented by a convenient length scale, and then draw vector B to the same scale with its tail starting from the tip of A, as shown in Figure. The **resultant vector** R = A + B is the vector drawn from the tail of A to the tip of B.

R is the vector drawn from the tail of the first vector to the tip of the last vector.

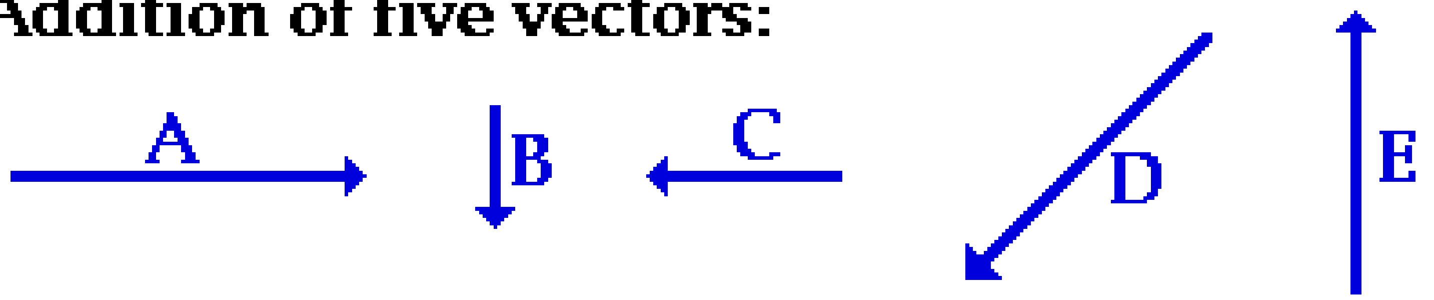






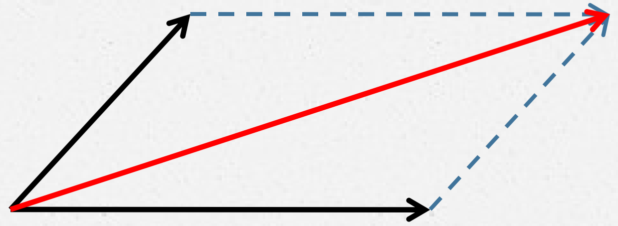
Move w over keeping the magnitude and direction the same.

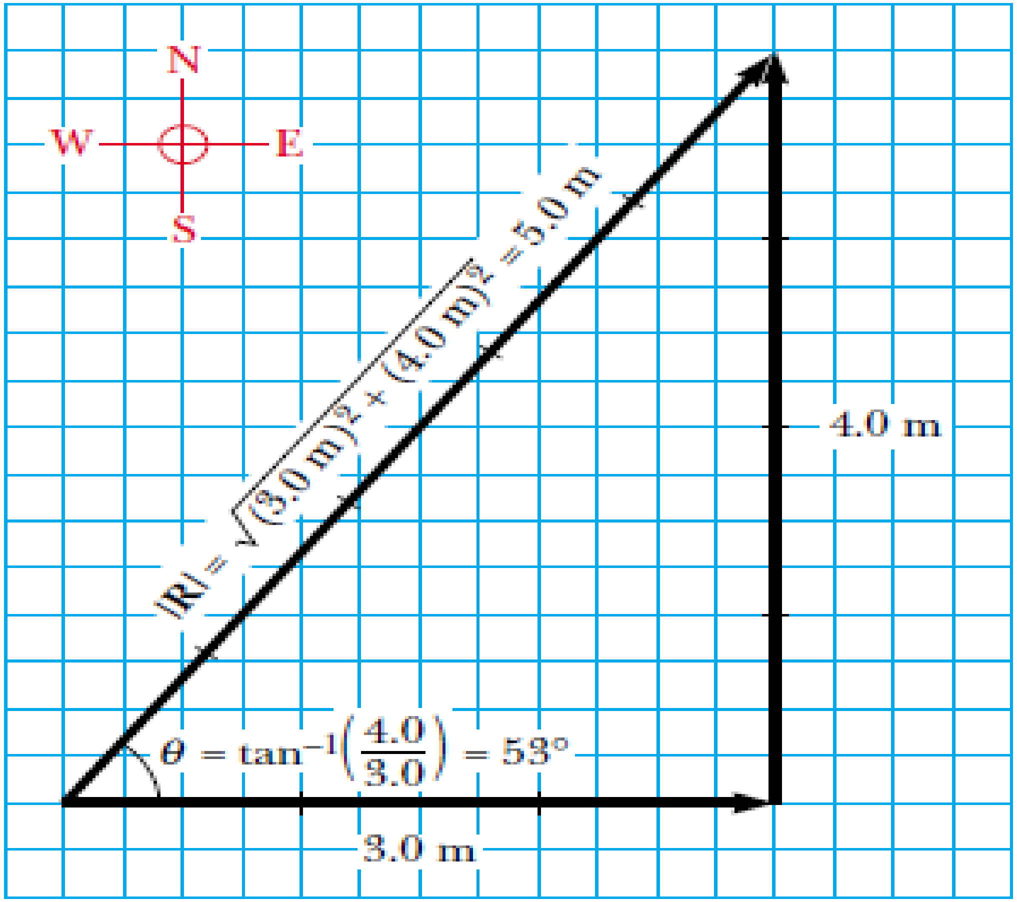
Addition of five vectors:



Parallelogram method

- When two vectors are joined tail to tail
- Complete the parallelogram
- The resultant is found by drawing the diagonal




N
W
E
S
|R| = √((3.0 m)² + (4.0 m)²) = 5.0 m
4.0 m
θ = tan⁻¹(4.0/3.0) = 53°
3.0 m

- **Negative of a Vector:**

The negative of the vector ***v*** is defined as the vector that when added to ***v*** gives zero for the vector sum. That is,

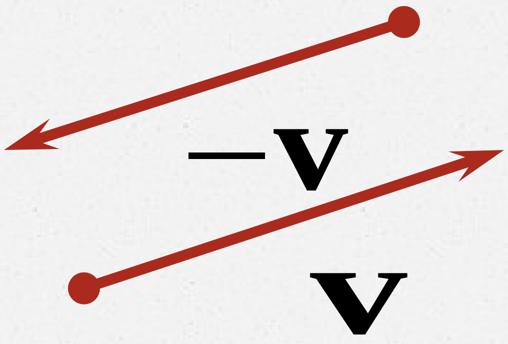
$$\boldsymbol{v} + (-\boldsymbol{v}) = 0.$$

The vectors ***v*** and - ***v*** have the same magnitude but point in opposite directions.

- **Subtracting Vectors:**

The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation A - B as vector -B added to vector A:

A – B = A + (-B)



- **Multiplying a Vector by a Scalar**

If vector A is multiplied by a positive scalar quantity *m*, then the product *m*A is a vector that has the same direction as A and magnitude *mA*.

If vector A is multiplied by a negative scalar quantity -*m*, then the product -*m*A is directed opposite A.

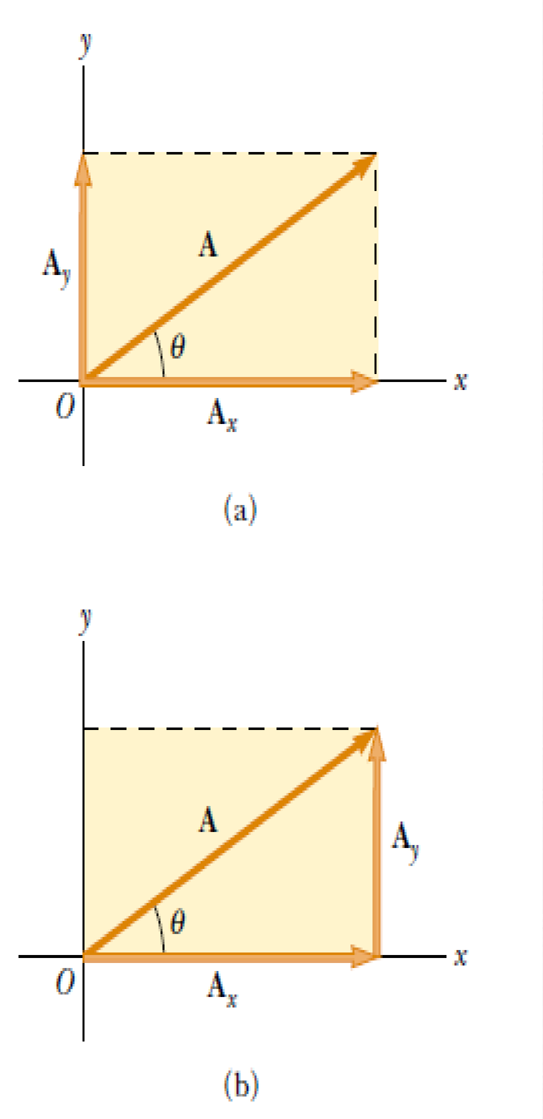
3.4 Components of a Vector

- The graphical method of adding vectors is not recommended whenever high accuracy is required or in three-dimensional problems.
- In this section, we describe a method of adding vectors that makes use of the projections of vectors along coordinate axes.
- These projections are called the components of the vector. Any vector can be completely described by its components.

- Consider a vector A lying in the *xy* plane and making an arbitrary angle θ with the positive *x* axis, as shown in Figure 3.13a
- This vector can be expressed as the sum of two other vectors A_x and A_y .

$$A = A_x + A_y$$

- The component A_x represents the projection of A along the *x* axis.
- The component A_y represents the projection of A along the *y* axis.
- These components can be positive or negative. The component A_x is positive if A_x points in the positive *x* direction and is negative if A_x points in the negative *x* direction. The same is true for the component A_y .



- The components of A are:

$$\sin\theta = \frac{A_y}{A} \qquad \cos\theta = \frac{A_x}{A}$$

$$\mathbf{A_x = A\cos\theta}$$

$$\mathbf{A_y = A\sin\theta}$$

- The magnitude of vector A is:

$$A = \sqrt{{A_x}^2 + {A_y}^2}$$

- The direction is:

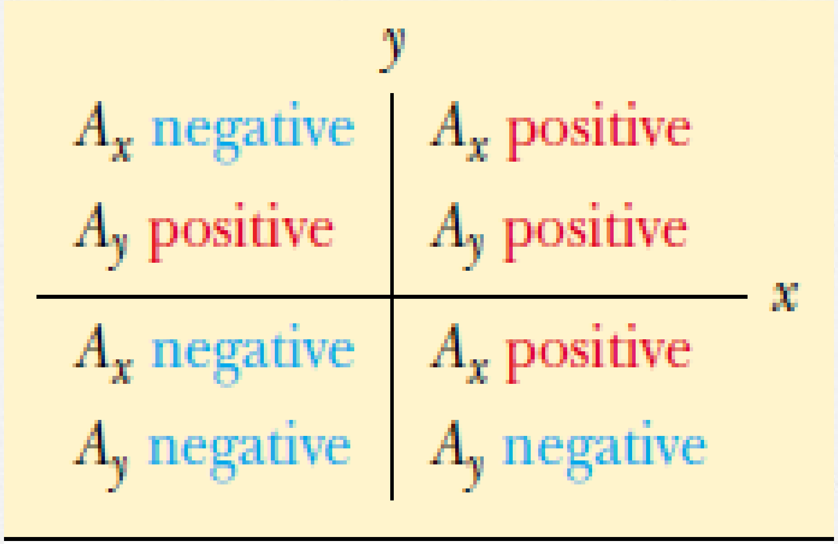
$$\theta = \tan^{-1}\frac{A_y}{A_x}$$

- Note that the signs of the components A_x and A_y depend on the angle θ.
- For example, if:

Θ= 120º , then A_x is negative and A_y is positive.

If

Θ= 225º , then A_x is negative and A_y is negative.



Unit Vectors

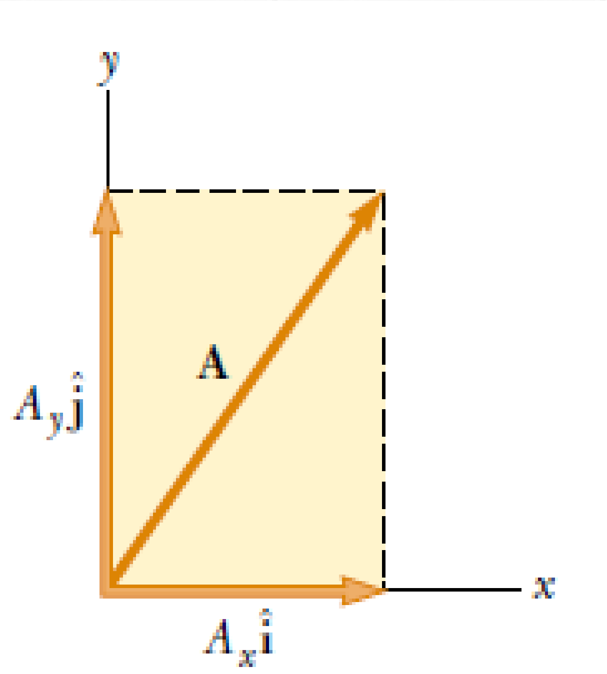
- Vector quantities often are expressed in terms of unit vectors.
- **A unit vector is a dimensionless vector having a magnitude of exactly 1.**
- Unit vectors are used to specify a given direction and have no other physical significance.
- We shall use the symbols $\hat{\imath}, \hat{\jmath}\ and\ \hat{k}$ to represent unit vectors pointing in the positive *x*, *y*, and *z* directions, respectively.
- The unit vectors form a set of mutually perpendicular vectors as shown in Figure 3.16a.

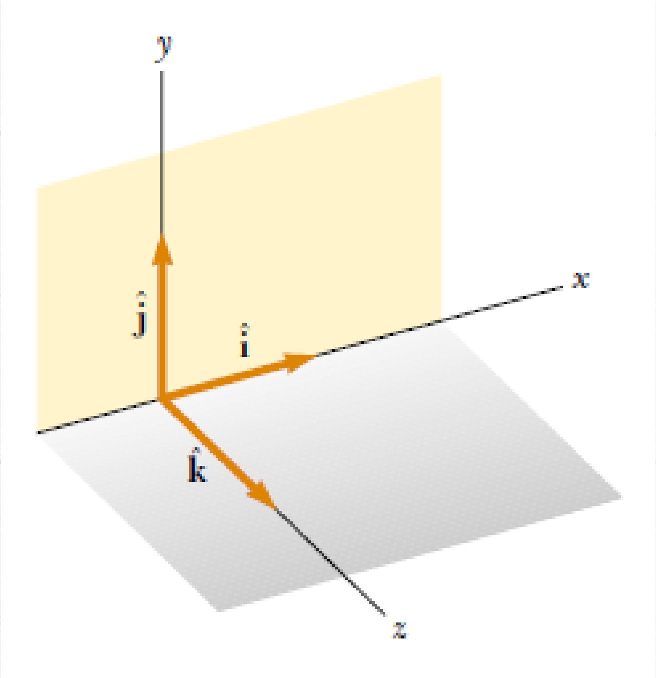
- The magnitude of each unit vector equals 1

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

- Thus, the unit–vector notation for the vector A is

$$A = A_x\hat{i} + A_y\hat{j}$$





- Now let us see how to use components to add vectors Suppose we wish to add vector B to vector A in last equation where vector B has components B_x and B_y.
- All we do is add the *x* and *y* components separately. The resultant vector R = A + B is therefore

$$\mathbf{R} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}})$$

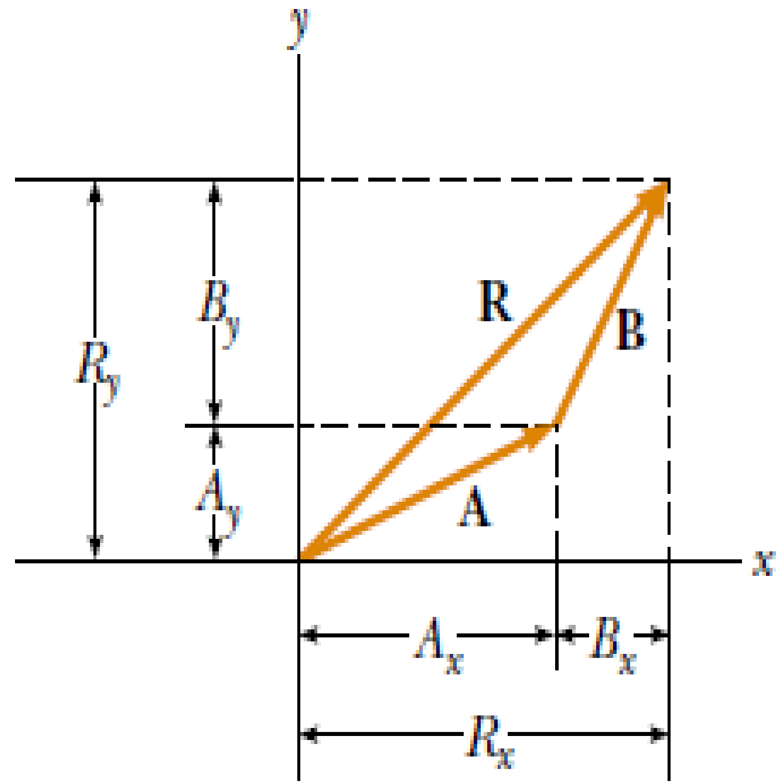
$$\mathbf{R} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}}$$

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x}$$



Example 3.3 The Sum of Two Vectors

Find the sum of two vectors **A** and **B** lying in the *xy* plane and given by

$$\mathbf{A} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\text{ m} \quad \text{and} \quad \mathbf{B} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})\text{ m}$$

Solution You may wish to draw the vectors to *conceptualize* the situation. We *categorize* this as a simple plug-in problem. Comparing this expression for **A** with the general expression $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$, we see that $A_x = 2.0$ m and $A_y = 2.0$ m. Likewise, $B_x = 2.0$ m and $B_y = -4.0$ m. We obtain the resultant vector **R**, using Equation 3.14:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (2.0 + 2.0)\hat{\mathbf{i}}\text{ m} + (2.0 - 4.0)\hat{\mathbf{j}}\text{ m}$$
$$= (4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}})\text{ m}$$

or

$$R_x = 4.0\text{ m} \qquad R_y = -2.0\text{ m}$$

The magnitude of **R** is found using Equation 3.16:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0\text{ m})^2 + (-2.0\text{ m})^2} = \sqrt{20}\text{ m}$$
$$= \boxed{4.5\text{ m}}$$

We can find the direction of **R** from Equation 3.17:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0\text{ m}}{4.0\text{ m}} = -0.50$$

Your calculator likely gives the answer $-27°$ for $\theta = \tan^{-1}(-0.50)$. This answer is correct if we interpret it to mean 27° clockwise from the *x* axis. Our standard form has been to quote the angles measured counterclockwise from the $+x$ axis, and that angle for this vector is $\boxed{\theta = 333°}$.

Example 3.4 The Resultant Displacement

A particle undergoes three consecutive displacements: $\mathbf{d}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})$ cm, $\mathbf{d}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})$ cm and $\mathbf{d}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})$ cm. Find the components of the resultant displacement and its magnitude.

Solution Three-dimensional displacements are more difficult to imagine than those in two dimensions, because the latter can be drawn on paper. For this problem, let us *conceptualize* that you start with your pencil at the origin of a piece of graph paper on which you have drawn x and y axes. Move your pencil 15 cm to the right along the x axis, then 30 cm upward along the y axis, and then 12 cm *vertically away* from the graph paper. This provides the displacement described by $\mathbf{d}_1$. From this point, move your pencil 23 cm to the right parallel to the x axis, 14 cm parallel to the graph paper in the $-y$ direction, and then 5.0 cm vertically downward toward the graph paper. You are now at the displacement from the origin described by $\mathbf{d}_1 + \mathbf{d}_2$. From this point, move your pencil 13 cm to the left in the $-x$ direction, and (finally!) 15 cm parallel to the graph paper along the y axis. Your final position is at a displacement $\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$ from the origin.

Despite the difficulty in conceptualizing in three dimensions, we can *categorize* this problem as a plug-in problem due to the careful bookkeeping methods that we have developed for vectors. The mathematical manipulation keeps track of this motion along the three perpendicular axes in an organized, compact way:

$$\mathbf{R} = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$$

$$= (15 + 23 - 13)\hat{\mathbf{i}}\text{ cm} + (30 - 14 + 15)\hat{\mathbf{j}}\text{ cm} + (12 - 5.0 + 0)\hat{\mathbf{k}}\text{ cm}$$

$$= (25\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 7.0\hat{\mathbf{k}})\text{ cm}$$

The resultant displacement has components $R_x = 25$ cm, $R_y = 31$ cm, and $R_z = 7.0$ cm. Its magnitude is

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$= \sqrt{(25\text{ cm})^2 + (31\text{ cm})^2 + (7.0\text{ cm})^2} = 40\text{ cm}$$

Example 3.5 Taking a Hike

Interactive

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.

Solution We *conceptualize* the problem by drawing a sketch as in Figure 3.19. If we denote the displacement vectors on the first and second days by **A** and **B**, respectively, and use the car as the origin of coordinates, we obtain the vectors shown in Figure 3.19. Drawing the resultant **R**, we can now *categorize* this as a problem we've solved before—an addition of two vectors. This should give you a hint of the power of categorization—many new problems are very similar to problems that we have already solved if we are careful to conceptualize them.

We will *analyze* this problem by using our new knowledge of vector components. Displacement **A** has a magnitude of 25.0 km and is directed 45.0° below the positive *x* axis. From Equations 3.8 and 3.9, its components are

$$A_x = A\cos(-45.0^\circ) = (25.0\text{ km})(0.707) = \boxed{17.7\text{ km}}$$

$$A_y = A\sin(-45.0^\circ) = (25.0\text{ km})(-0.707) = \boxed{-17.7\text{ km}}$$

The negative value of A_y indicates that the hiker walks in the negative *y* direction on the first day. The signs of A_x and A_y also are evident from Figure 3.19.

The second displacement **B** has a magnitude of 40.0 km and is 60.0° north of east. Its components are

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = \boxed{20.0\text{ km}}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = \boxed{34.6\text{ km}}$$

(B) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.

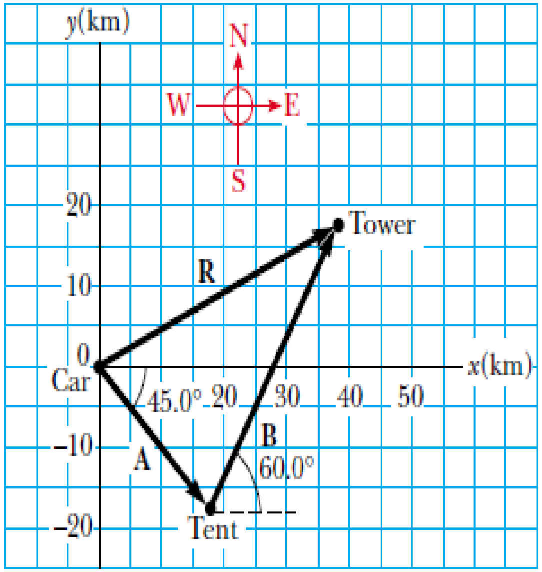


Figure 3.19 (Example 3.5) The total displacement of the hiker is the vector **R** = **A** + **B**.

Solution The resultant displacement for the trip **R** = **A** + **B** has components given by Equation 3.15:

$$R_x = A_x + B_x = 17.7\text{ km} + 20.0\text{ km} = \boxed{37.7\text{ km}}$$

$$R_y = A_y + B_y = -17.7\text{ km} + 34.6\text{ km} = \boxed{16.9\text{ km}}$$

In unit–vector form, we can write the total displacement as

$$\mathbf{R} = \boxed{(37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}})\text{ km}}$$

Using Equations 3.16 and 3.17, we find that the vector **R** has a magnitude of 41.3 km and is directed 24.1° north of east.

Let us *finalize.* The units of **R** are km, which is reasonable for a displacement. Looking at the graphical representation in Figure 3.19, we estimate that the final position of the hiker is at about (38 km, 17 km) which is consistent with the components of **R** in our final result. Also, both components of **R** are positive, putting the final position in the first quadrant of the coordinate system, which is also consistent with Figure 3.19.

Example 3.6 Let's Fly Away!

A commuter airplane takes the route shown in Figure 3.20. First, it flies from the origin of the coordinate system shown to city A, located 175 km in a direction 30.0° north of east. Next, it flies 153 km 20.0° west of north to city B. Finally, it flies 195 km due west to city C. Find the location of city C relative to the origin.

Solution Once again, a drawing such as Figure 3.20 allows us to *conceptualize* the problem. It is convenient to choose the coordinate system shown in Figure 3.20, where the *x* axis points to the east and the *y* axis points to the north. Let us denote the three consecutive displacements by the vectors **a**, **b**, and **c**.

We can now *categorize* this problem as being similar to Example 3.5 that we have already solved. There are two primary differences. First, we are adding three vectors instead of two. Second, Example 3.5 guided us by first asking for the components in part (A). The current Example has no such guidance and simply asks for a result. We need to *analyze* the situation and choose a path. We will follow the same pattern that we did in Example 3.5, beginning with finding the components of the three vectors **a**, **b**, and **c**. Displacement **a** has a magnitude of 175 km and the components

$$a_x = a\cos(30.0°) = (175\text{ km})(0.866) = 152\text{ km}$$

$$a_y = a\sin(30.0°) = (175\text{ km})(0.500) = 87.5\text{ km}$$

Displacement **b**, whose magnitude is 153 km, has the components

$$b_x = b\cos(110°) = (153\text{ km})(-0.342) = -52.3\text{ km}$$

$$b_y = b\sin(110°) = (153\text{ km})(0.940) = 144\text{ km}$$

Finally, displacement **c**, whose magnitude is 195 km, has the components

$$c_x = c\cos(180°) = (195\text{ km})(-1) = -195\text{ km}$$

$$c_y = c\sin(180°) = 0$$

Therefore, the components of the position vector **R** from the starting point to city C are

$$R_x = a_x + b_x + c_x = 152\text{ km} - 52.3\text{ km} - 195\text{ km}$$

$$= -95.3\text{ km}$$

$$R_y = a_y + b_y + c_y = 87.5\text{ km} + 144\text{ km} + 0 = 232\text{ km}$$

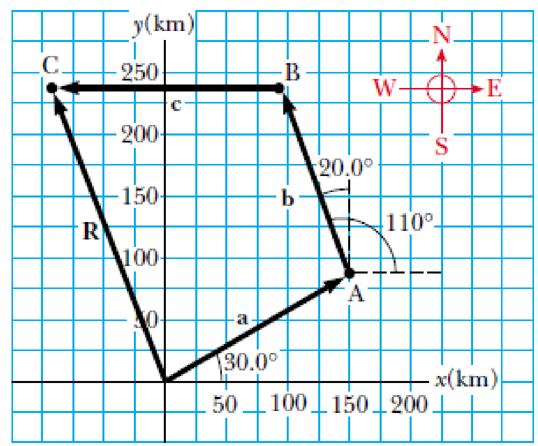


Figure 3.20 (Example 3.6) The airplane starts at the origin, flies first to city A, then to city B, and finally to city C.

Vectors product

Scalar (dot)

- $A \cdot B = |A||B| \cos\theta$
- The result of the product is scalar
- The result equal zero if:

  $\theta = 90° \; or \; 270°$
- $i \cdot i = j \cdot j = k \cdot k = 1$
- $i \cdot j = j \cdot k = k \cdot i = 0$

Vector (cross)

- $A \times B = |A||B| \sin\theta$
- The result of the product is vector.
- The result equal zero if:
- $\theta = 0° \; or \; 180°$
- $i \times i = j \times j = k \times k = 0$

  $i \times j = k, \qquad j \times i = -k$

  $j \times k = i, \qquad k \times j = -i$

  $k \times i = j, \qquad i \times k = -j$

Derivation

- How do we show that $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$
- Start with $$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$ $$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
- Then

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

- But $$\hat{i}\cdot\hat{j} = 0;\ \hat{i}\cdot\hat{k} = 0;\ \hat{j}\cdot\hat{k} = 0$$ $$\hat{i}\cdot\hat{i} = 1;\ \hat{j}\cdot\hat{j} = 1;\ \hat{k}\cdot\hat{k} = 1$$
- So $$\vec{A}\cdot\vec{B} = A_x\hat{i}\cdot B_x\hat{i} + A_y\hat{j}\cdot B_y\hat{j} + A_z\hat{k}\cdot B_z\hat{k}$$ $$= A_x B_x + A_y B_y + A_z B_z$$

Cross product

- Task:

By the same way find $A \times B$ for last two vectors.

Perpendicular Vectors

- Suppose there are two vectors **a** and **b,** vector **a** is *perpendicular* or *orthogonal* to $\boldsymbol{b}$, so the angle between them is $\theta = \pi/2 = 90$.
- The scalar product is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0$$

- Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$.

Example

- If the vectors **a** and **b** have lengths 2m and 3m, and the angle between them is 60, find $\mathbf{a} \cdot \mathbf{b}$.

Solution

According to the definition,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\ \mathit{cos(60)} = 2 \times 3 \times \tfrac{1}{2} = 3\text{m}$$

Example

- Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathrm{i} - 4\mathbf{j} + 2\mathbf{k}$.

Solution

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(5\mathrm{i} - \mathbf{4j} + 2\mathbf{k}) =$$

$$2(5) + 2(-4) + (-1)(2) = 0,$$

these vectors are perpendicular.

Example

- Find the angle between **a**= 2i +2j-k and **b** =5i-3j+2k

Solution

Let θ be the required angle. Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \text{ and}$$

$$|\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a}\cdot\mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2,$$

the definition of dot product gives

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

- So the angle between a and b is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84^\circ)$$