MIDTERM (2) Solution Guide

Question 1

(i) Show $\lim_{(x,y)\to(0,0)} \dfrac{x^3+4x^2+2y^2}{2x^2+y^2}$ exists. (2 Marks)

path (1) $x = 0$

$$\lim_{y\to 0} \frac{2y^2}{y^2} \quad = \quad 2$$

path (2) $y = 0$

$$\lim_{x\to 0} \frac{x^3+4x^2}{2x^2} = 2$$

If the limit exists, it must be equal to 2. Using Theorem 2.1

$$\begin{aligned} |f(x,y) - L| &= \left| \frac{x^3+4x^2+2y^2}{2x^2+y^2} - 2 \right| \\ &= \left| \frac{x^3}{2x^2+y^2} \right| \\ &\leq \left| \frac{x^3}{2x^2} \right| \\ &= \left| \frac{x}{2} \right|. \end{aligned}$$

Since $\lim_{x\to 0} \left|\frac{x}{2}\right| = 0$, Theorem 2.1 gives $\lim_{(x,y)\to(0,0)} \dfrac{x^3+4x^2+2y^2}{2x^2+y^2} = 2$.

(ii) Let $f(x,y) = e^{xy} + \ln(x^2+y)$, show that $f_{xy} = f_{yx}$. (3 Marks)

$$\begin{aligned} f_x &= ye^{xy} + \frac{2x}{x^2+y} \\ f_{xy} &= e^{xy} + xye^{xy} - \frac{2x}{(x^2+y)^2} \\ f_y &= xe^{xy} + \frac{1}{x^2+y} \\ f_{yx} &= e^{xy} + xye^{xy} - \frac{2x}{(x^2+y)^2} \, . \end{aligned}$$

So $f_{xy} = f_{yx}$

(iii) Find equations of the tangent plane and the normal line to

$$z = 6 - x^2 - y^2 \quad \text{at the point } (1, 2, 1).$$

(2 Marks)

$$f(x,y) = 6 - x^2 - y^2 \quad f(1,2) = 6 - 1 - 4 = 1$$

$$f_x = -2x \quad f_x(1,2) = -2$$

$$f_y = -2y \quad f_y(1,2) = -4$$

- A normal vector is $\langle -2, -4, -1 \rangle$
- An equation of the tangent plane is

$$\begin{aligned} z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ z &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 1 - 2(x-1) - 4(y-2) \end{aligned}$$

- The equations of the normal line are

$$x = 1 - 2t, \quad y = 2 - 4t, \quad z = 1 - t \; .$$

(v) Consider the function

$$z = f(x,y) = \sin(x+y) \quad \text{with} \quad x = uv^2 \quad \text{and} \quad y = u^2 + \frac{1}{v},$$

find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. (3 Marks)

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \cos(x+y)v^2 + \cos(x+y)2u \\
&= \cos(uv^2 + u^2 + \frac{1}{v})(v^2 + 2u)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \cos(x+y)2uv + \cos(x+y)\,\frac{-1}{v^2} \\
&= \cos(uv^2 + u^2 + \frac{1}{v})(2uv - \frac{1}{v^2})\ .
\end{aligned}$$

Question 2

(i) For $f(x,y) = x^2 + y^2$,

compute $D_u f(1,-1)$ for $\hat{u}$ in the direction of $\overrightarrow{v} = \langle -3, 4\rangle$ (2 Marks)

$$D_u f(x,y) = \nabla f(x,y) \cdot \hat{u}$$
$$D_u f(1,-1) = \nabla f(1,-1) \cdot \hat{u} \ .$$

Now

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y\rangle$$
$$\nabla f(1,-1) = \langle 2, -2\rangle \ .$$

So

$$\begin{aligned} D_u f(1,-1) &= \langle 2,-2\rangle \cdot \langle -3/5, 4/5\rangle \\ &= (2)(-3/5) + (-2)(/54) = -14/5 \end{aligned}$$

(ii) locate all critical points and classify them using (Second Derivatives Test)

$$f(x,y) = e^{-x^2}(y^2+1)$$

(3 Marks)

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$$f_x = -2xe^{-x^2}(y^2+1) \quad (1)$$
$$f_y = 2e^{-x^2}y \quad (2)$$

- Solving (1) and (2) to find critical points

$$-2xe^{-x^2}(y^2+1) = 0$$
$$2e^{-x^2}y = 0 \Longrightarrow y = 0$$

when $y = 0 \Longrightarrow x = 0$. So the critical point is $(0,0)$.

- To classify the C.P

$$\begin{aligned} f_{xx} &= 4x^2e^{-x^2}(y^2+1) - 2e^{-x^2}(y^2+1) = (4x^2-2)e^{-x^2}(y^2+1) \\ f_{yy} &= 2e^{-x^2} \\ f_{xy} &= -4xye^{-x^2} \end{aligned}$$

so

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 \\ &= (-2)(2) - (0) = -4 < 0\ . \end{aligned}$$

So f has a saddle pint at $(0,0)$.

(iii) Find the volume beneath the surface and above the rectangular region

$$z = x^2 + y^2, \quad 0 \le x \le 3, \quad 1 \le y \le 4$$

(3 Marks)

$$\begin{aligned} \int_{y=1}^{y=4}\int_{x=0}^{x=3}(x^2+y^2)\,dx\,dy &= \int_{y=1}^{y=4}\left[\frac{x^3}{3}+y^2x\right]_{x=0}^{x=3}dy \\ &= \int_{y=1}^{y=4}\left[\frac{27}{3}+3y^2\right]dy \\ &= \int_{y=1}^{y=4}[9+3y^2]\,dy \\ &= [9y+y^3]_{y=1}^{y=4} = 90\ . \end{aligned}$$

(v) Change the order of integration

$$\int_{x=0}^{x=1}\int_{y=0}^{y=2x} f(x,y)\,dy\,dx = \int_{y=0}^{y=2}\int_{x=y/2}^{x=1} f(x,y)\,dx\,dy$$

(2 Marks)