MIDTERM (1) Solution Guide

Question 1

(i) Determine whether the given pair of vectors is parallel $\overrightarrow{a} = \langle 1, -2, 5\rangle$ and $\overrightarrow{b} = \langle 3, -6, 15\rangle$. (2 Marks)

Vectors are parallel if they are scalar multiple i.e.

$$\overrightarrow{b} = c\overrightarrow{a}\ .$$

Now

$$\begin{aligned} \langle 3, -6, 15\rangle &= 3\langle 1, -2, 5\rangle \\ \overrightarrow{b} &= 3\overrightarrow{a} \end{aligned}$$

So, $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel

(ii) Show that the two vectors $\overrightarrow{a} = 3\hat{i}$ and $\overrightarrow{b} = 6\hat{j} - 2\hat{k}$ are orthogonal. (2 Marks)

$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{b} &= \langle 3, 0, 0\rangle \cdot \langle 0, 6, -2\rangle \\ &= (3)(0) + (0)(6) + 0(-2) = 0 \end{aligned}$$

(iii) Find the angle between the vectors $\overrightarrow{a} = \langle 0, -2, 3\rangle$ and $\overrightarrow{b} = \langle 1, 1, 2\rangle$. (3 Marks)

$$
\begin{aligned}
\cos\theta &= \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{\|\overrightarrow{a}\|\cdot\|\overrightarrow{b}\|} \\
&= \frac{(0)(1)+(-2)(1)+(3)(2)}{\sqrt{0+4+9}\cdot\sqrt{1+1+4}} \\
&= \frac{4}{\sqrt{13}\cdot\sqrt{6}} \\
&= \frac{4}{\sqrt{78}} \\
\theta &= \cos^{-1}\frac{4}{\sqrt{78}} \\
&= \cos^{-1}\frac{4\sqrt{78}}{78} \\
&= \cos^{-1}\frac{2\sqrt{78}}{39} \\
&\simeq 63.1\ .
\end{aligned}
$$

(v) Find equations for the line passing through the points $P(1,2,-1)$ and $Q(5,-3,4)$. (3 Marks) First, a vector that is parallel to the line is

$$\overrightarrow{PQ} = \langle 5-1, -3-2, 4+1\rangle = \langle 4,-5,5\rangle\ .$$

So that parametric equations for the line are

$$x-1=4t, \quad y-2=-5t, \quad \text{and} \quad z+1=5t$$

and symmetric equations of the line are

$$\frac{x-1}{4} = \frac{y-2}{-5} = \frac{z+1}{5}$$

Question 2

(i) Find the velocity and position of an object at any time t, given that its acceleration is

$$\vec{a}(t) = e^t\,\hat{i} + e^{-t}\,\hat{k}$$

its initial velocity is $\vec{v}(0) = \hat{i} + 2\hat{j}$ and its initial position is $\vec{r}(0) = 3\hat{i} + \hat{j} + 2\hat{k}$. (4 Marks)

Since $\vec{a}(t) = \vec{v'}(t)$, we integrate once to obtain

$$\begin{aligned}
\vec{v}(t) &= \int \vec{a}(t)dt \\
&= \int [e^t\,\hat{i} + e^{-t}\,\hat{k}]dt \\
&= e^t\,\hat{i} - e^{-t}\,\hat{k} + c_1 \\
\vec{v}(0) &= \hat{i} - \hat{k} + c_1
\end{aligned}$$

We use the given initial velocity, to determine the value of c_1

$$\begin{aligned}
\hat{i} + 2\hat{j} &= \hat{i} - \hat{k} + c_1 \\
c_1 &= \hat{i} + 2\hat{j} - \hat{i} + \hat{k} \\
&= 2\hat{j} + \hat{k}
\end{aligned}$$

so this gives us the velocity

$$\begin{aligned}
\vec{v}(t) &= e^t\,\hat{i} - e^{-t}\,\hat{k} + 2\hat{j} + \hat{k} \\
&= e^t\,\hat{i} + 2\hat{j} + (1 - e^{-t})\hat{k}
\end{aligned}$$

Since $\vec{v}(t) = \vec{r'}(t)$, we integrate again, to obtain

$$\begin{aligned}
\vec{r}(t) &= \int \vec{v}(t)dt \\
&= \int [e^t\,\hat{i} + 2\hat{j} + (1 - e^{-t})\hat{k}]dt \\
&= e^t\,\hat{i} + 2t\hat{j} + (t + e^{-t})\hat{k} + c_2 \\
\vec{r}(0) &= \hat{i} + \hat{k} + c_2.
\end{aligned}$$

We can use the given initial position to determine the value of c_2

$$\begin{aligned} 3\hat{i} + \hat{j} + 2\hat{k} &= \hat{i} + \hat{k} + c_2 \\ c_2 &= 3\hat{i} + \hat{j} + 2\hat{k} - \hat{i} - \hat{k} \\ &= 2\hat{i} + \hat{j} + \hat{k}. \end{aligned}$$

This gives us the position vector

$$\overrightarrow{r}(t) = (e^t + 2)\,\hat{i} + (2t+1)\hat{j} + (t + 1 + e^{-t})\hat{k}.$$

(ii) Find unit tangent vector $\overrightarrow{T}(t)$, unit normal vectors $\overrightarrow{N}(t)$ and the curvature κ to the curve defined by

$$\overrightarrow{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}.$$

(6 Marks)

$$\overrightarrow{r'}(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}.$$

and

$$\begin{aligned} \left\|\overrightarrow{r'}(t)\right\| &= \sqrt{\sin^2 t + \cos^2 t + 1} \\ &= \sqrt{2}\,. \end{aligned}$$

The unit tangent vector $\overrightarrow{T}(t)$ is

$$\begin{aligned} \overrightarrow{T}(t) &= \frac{\overrightarrow{r'}(t)}{\left\|\overrightarrow{r'}(t)\right\|} \\ &= \frac{-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\left(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}\right). \end{aligned}$$

To find the unit normal vectors $\overrightarrow{N}(t)$

$$\overrightarrow{T'}(t) = \frac{1}{\sqrt{2}}\left(-\cos t\,\hat{i} - \sin t\,\hat{j}\right)$$

and

$$\left\|\overrightarrow{T'}(t)\right\| = \frac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}.$$

Now

$$\begin{aligned}\overrightarrow{N}(t) &= \frac{\overrightarrow{T'}(t)}{\left\|\overrightarrow{T'}(t)\right\|} \\ &= \frac{\frac{1}{\sqrt{2}}\left(-\cos t\,\hat{i}-\sin t\,\hat{j}\right)}{\frac{1}{\sqrt{2}}} = -\cos t\,\hat{i}-\sin t\,\hat{j}.\end{aligned}$$

Finally, the curvature is

$$\frac{\left\|\overrightarrow{T'}(t)\right\|}{\left\|\overrightarrow{r'}(t)\right\|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}.$$