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Course	**Financial Mathematics**
Unit course	**FIN 118**
Number Unit	**9**
Unit Subject	**Compound Interest** **Non annual Compound Interest** **Continuous Compound Interest**

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!!! remember what we saw last time

- ✓ The relationship between time and money.
- ✓ The simple interest rate and the interest amount
- ✓ The present value of one future cash flow
- ✓ The future value of an amount borrowed or invested.
- ✓ The relationship between Real Interest Rate, Nominal Interest Rate and Inflation.

we will see in this unit

- ✓The compound interest rate and the interest amount
- ✓ How to Calculate the future value of a single sum of money invested today for several periods.
- ✓ How to Calculate the interest rate or the number of periods or the principal that achieve a fixed future value.

LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand compound interest, including accumulating, discounting and making comparisons using the effective interest rate.

2. Distinguish between compound interest.

3. Identify variables fundamental to solving interest problems.

4. Solve problems including future and present value.

The Compound Interest

Definition1: In each subsequent period, the interest amount computed is used to form a new principal sum, which is used to compute the next interest due.

* As we said, Compound Interest uses the Sum of Principal & Interest as a base on which to calculate new Interest and new Principal !

$$
\begin{aligned}
FV_1 &= PV(1+i_1) && \textit{one period} \\
FV_2 &= FV_1(1+i_2) = PV(1+i_1)(1+i_2) && \textit{two periods} \\
FV_3 &= FV_2(1+i_3) = PV(1+i_1)(1+i_2)(1+i_3) && \textit{three periods} \\
&\ldots \\
FV_n &= PV(1+i_1)(1+i_2)(1+i_3)\cdots(1+i_n) && \textit{n periods}
\end{aligned}
$$

The Compound Interest

Definition2: If the interest rate is constant over different periods we have: $i_1 = i_2 = i_3 = \ldots. = i_n = i$

and

FV_n = Principal ×(1+ Interest Rate)$^{\text{number of Periods}}$

$$\boxed{FV_n = PV(1+i)^n}$$

$$\boxed{PV = \frac{FV_n}{(1+i)^n}}$$

$$\boxed{i = \left[\frac{FV_n}{PV}\right]^{\frac{1}{n}} - 1}$$

$$\boxed{n = \frac{\ln(FV_n / PV)}{\ln(1+i)}}$$

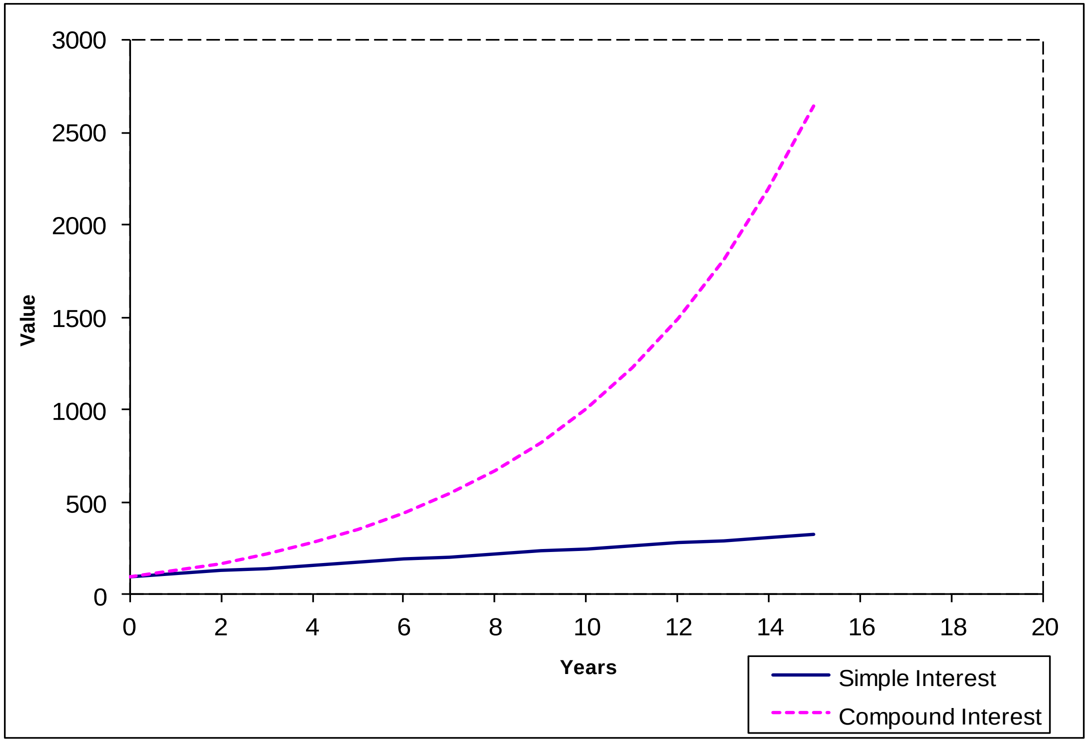
!!! Remember

$$\boxed{FV_n = PV + I}$$

The Compound Interest

Property1:

The compound interest rate is a geometric sequences but the simple interest is an arithmetic sequences.



Simple interest: Linear growth i = 0.15
Compound interest: Geometric growth   i = 0.15

The Compound Interest

More Examples

Example1: Future Value and Interest

How much money would you pay in interest if you borrowed \$1600 for 3 years at 16% compound interest per annum?

Solution:

Convert the percent to a decimal: 16% = 0.16

$$\boxed{FV_3 = PV(1+i)^3 = 1600(1+0.16)^3 = 2497.434}$$

$$\boxed{FV_n = PV + I \Rightarrow I = FV_n - PV}$$

$$\boxed{I = 2497.434 - 1600 = 897.4336}$$

The Compound Interest

More Examples

Example2: Present Value

What is the present value of $150000 to be received 5 years from today if the discount rate (annual compounded interest) is 10%?

Solution:

Convert the percent to a decimal: 10% = 0.1

$$PV = \frac{FV_5}{(1+i)^5} = \frac{150000}{(1+0.1)^5} = \$93138.198$$

The Compound Interest

More Examples

Example3: Interest rate

Assume that the initial amount to invest is PV = \$100 and the interest rate is constant over time. What is the compound interest rate in order to have \$150 after 5 years?

Solution:

PV = \$100 and FV_5 = \$150

$$FV_5 = PV(1+i)^5 \Rightarrow 1+i = \left[\frac{FV_5}{PV}\right]^{\frac{1}{5}} \Rightarrow i = \left[\frac{FV_5}{PV}\right]^{\frac{1}{5}} - 1$$

$$i = \left[\frac{150}{100}\right]^{\frac{1}{5}} - 1 = 1.084 - 1 = 0.084$$

$$i = 8.4\%$$

The Compound Interest

More Examples

Example4: The number of periods (n)

Find the number of periods to double your investment at 6% compound interest per annum .

Solution:

PV = x and FV_n = 2x

$$FV_n = PV(1+i)^n \Rightarrow (1+i)^n = \left[\frac{FV_n}{PV}\right] \Rightarrow n\ln(1+i) = \ln(FV_n/PV)$$

$$n = \frac{\ln(FV_n/PV)}{\ln(1+i)} = \frac{\ln(2x/x)}{\ln(1+0.06)} = \frac{\ln(2)}{\ln(1.06)} = 11.895\ years$$

Convert the result:

11 years + 0.895 × 12 months= 11 years + 10.74 months

11 years + 10 months + 0.74 × 30 days

n = 11 years + 10 months + 22 days

Question 1 ? How to calculate the FV if we have more than one compounding periods per year ?

Response:

The table shows some common compounding periods and how many times per year interest is paid for them.

Compounding Periods	Times per year (t)
Annually	1
Semi-annually	2
Quarterly	4
Monthly	12

And

$$\boxed{FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t}}$$

!!! If t=1 we retrieve the old formula

Non annual Compound Interest

Example1 : Future Value semi-annually

You invested $1800 in a savings account that pays 4.5% interest compounded semi-annually. Find the value of the investment in 12 years.

Solution:

Convert the percent to a decimal: 4.5% = 0.045

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t}$$

$$FV_{12,2} = 1800\left(1+\frac{0.045}{2}\right)^{12\times 2}$$

$$= 1800(1 + 0.0225)^{24}$$

$$= 1800(1.0225)^{24} = \$3070.38$$

Non annual Compound Interest

Example2: Future Value Quarterly

You invested $3700 in a savings account that pays 2.5% interest compounded quarterly. Find the value of the investment in 10 years.

Solution:

Convert the percent to a decimal: 2.5% = 0.025

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t}$$

$$FV_{10,4} = 3700\left(1+\frac{0.025}{4}\right)^{10\times 4}$$

$$= 3700(1 + 0.00625)^{40}$$

$$= 3700(1.00625)^{40} = \$4747.2$$

Non annual Compound Interest

Example3: Future Value monthly

You invested $1700 in a savings account that pays 1.5% interest compounded monthly. Find the value of the investment in 15 years.

Solution:

Convert the percent to a decimal: 1.5% = 0.015

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t}$$

$$FV_{15,12} = 1700\left(1+\frac{0.015}{12}\right)^{15\times 12}$$

$$= 1700(1 + 0.00125)^{180}$$

$$= 1700(1.00125)^{180} = \$2128.65$$

Non annual Compound Interest

Example4: Present Value

You expect to need $1500 in 3 years. Your bank offers 4% interest compounded semiannually. How much money must you put in the bank today (PV) to reach your goal in 3 years?

Solution:

Convert the percent to a decimal: 4% = 0.04

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t} \Rightarrow PV = \frac{FV_{n,t}}{\left(1+\frac{i}{t}\right)^{n\times t}}$$

$$PV = \frac{1500}{\left(1+\frac{0.04}{2}\right)^{3\times 2}} = \frac{1500}{(1.02)^6} = \$1331.96$$

Non annual Compound Interest

Example5: Making a choice

Suppose a bank quotes nominal annual interest rates on five-year of:

* 6.6% compounded annually,
* 6.5% compounded semi-annually, and
* 6.4% compounded monthly.

Which rate should an investor choose for an investment of $10000?

Solution:

Convert the percent to a decimal:

6.6% = 0.066; 6.5% = 0.065 and 6.4% = 0.064

The times per year is respectively t= 1; t = 2 and

t = 12.

Non annual Compound Interest

Solution : continued

$$\boxed{FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t}}$$

First proposition: $$FV_{5,1} = 10000\left(1+\frac{0.066}{1}\right)^{5\times 1} = 13765.31$$

Second proposition: $$FV_{5,2} = 10000\left(1+\frac{0.065}{2}\right)^{5\times 2} = 13768.94$$

Third proposition: $$FV_{5,12} = 10000\left(1+\frac{0.064}{12}\right)^{5\times 12} = 13759.57$$

We choose the second proposition.
6.5% compounded semi-annually provides the best return on investment.

Question 2 ? What would happen to our money if we compounded a really large number of times?

Response:

We would have to compound not just every hour, or every minute or every second but for every millisecond. We have:

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t} \xrightarrow{t\to\infty} PV \times e^{n\times i}$$

* Then with Continuous compounding interest we have:

$$\boxed{FV_{n,t} = PV \times e^{n\times i}}$$

Continuous Compound Interest

Example1: Future Value

If you invest \$1000 at an annual interest rate of 5% compounded continuously, calculate the final amount you will have in the account after five years.

Solution:

Convert the percent to a decimal 5% =0.05

With continuous compounding formula we obtain

$$FV_5 = PV \times e^{n \times i} = 1000 \times e^{5 \times 0.05} = \$1284.02$$

Continuous Compound Interest

Example2: Finding the time

How long will it take an investment of $10000 to grow to $15000 if it is invested at 9% compounded continuously?

Solution:

Convert the percent to a decimal 9% =0.09

With continuous compounding formula we obtain

$$FV_n = PV \times e^{n\times i} \Rightarrow e^{n\times i} = \frac{FV_n}{PV}$$

$$\Rightarrow n\times i = \ln\left(FV_n / PV\right) \Rightarrow n = \frac{\ln\left(FV_n / PV\right)}{i}$$

$$\Rightarrow n = \frac{\ln(1.5)}{0.09} = 4.505 \text{ years}$$

= 4 years + 6 months + 2 days

Continuous Compound Interest

Example3: Finding the interest rate

What is the interest rate compounded continuously of an investment of $10000 to grow to $20000 if it is invested for 7 years?

Solution:

$$FV_n = PV \times e^{n\times i} \Rightarrow e^{n\times i} = \frac{FV_n}{PV}$$

$$\Rightarrow n\times i = \ln\left(FV_n / PV\right) \Rightarrow i = \frac{\ln\left(FV_n / PV\right)}{n}$$

$$\Rightarrow i = \frac{\ln(2)}{7} = 0.099 \Rightarrow i = 9.9\%$$

Compound Interest

Example4: Making a choice

What amount will an account have after 5 years if \$100 is invested at an annual nominal rate of 8% compounded annually? Semiannually? continuously?

Solution:

Compounded annually: $FV_5 = 100 \times (1+0.08)^5 = 146.93$

Compounded semi-annually: $FV_{5,2} = 100 \times \left(1 + \frac{0.08}{2}\right)^{5\times 2}$

$= 148.02$

Compounded continuously: $FV_5 = 100 \times e^{0.08\times 5} = 149.18$

We choose the third proposition.
8% compounded continuously provides the best return on investment.

It's time to review

Simple Interest	Compound interest
$I = PV \times i \times n$	
$FV_n = PV + I$	$FV_n = PV + I$
$FV_n = PV(1 + i \times n)$	$FV_n = PV(1+i)^n$
More than one compounding periods per year	Continuous Compound Interest
$FV_{n,t} = PV\left(1 + \frac{i}{t}\right)^{n \times t}$	$FV_n = PV \times e^{n \times i}$

Real Interest Rate = Nominal Interest Rate - Inflation

we will see in the next unit

- ✓ Meant of simple Annuity

- ✓Simple Annuity: Ordinary Annuity,

Annuity Due (unit10)