Al-Imam Muhammad Ibn Saud Islamic University
College of Economics and Administration Sciences
Department of Finance and Investment

جامعة الإمام محمد بن سعود الإسلامية
كلية الاقتصاد والعلوم الإدارية
قسم التمويل والاستثمار

Course	**Financial Mathematics**
Unit course	**FIN 118**
Number Unit	**5**
Unit Subject	**Matrices**

Dr. Lotfi Ben Jedidia
Dr. Imed Medhioub

we will see in this unit

- ✓ Matrix / Matrices
- ✓ Different types of matrices
- ✓ Usual operations on matrices

LEARNING OUTCOMES

At the end of this chapter, you should be able to:

1. Define and use the matrix form.

2. Distinct between the different types of matrices and its properties.

3. Make usual operations on matrices.

Matrix / Matrices

Definition:

In Mathematics, matrices are used to store information.

- This information is written in a rectangular arrangement of rows and columns.
- Each entry, or element, of a matrix has a precise position and meaning.

Example1:

4 Columns

3 Rows

$$A_{(3,4)} = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 0 & 2 & 1 \\ 5 & 1 & 1 & 7 \end{bmatrix}$$

Example of Matrix

- Food shopping online: people go online to shop four items and have them delivered to their homes.
- Cartons of eggs, bread, bags of rice, packets of chickens were ordered online and the people left their address for delivery.
- A selection of orders may look like this:

Order / Address	Carton of eggs	Bread	Rice	Chicken
Al Wuroud	2	1	3	0
Al Falah	4	0	2	1
Al Izdihar	5	1	1	7

Matrix notation

A matrix is defined by its order which is always number of rows by number of columns.

$$A_{(m,n)} = \begin{bmatrix} a_{11} & a_{12} & a_{13}\dots & a_{1n} \\ a_{21} & a_{22} & a_{23}\dots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & a_{m3}\dots & a_{mn} \end{bmatrix}$$

- A horizontal set of elements is called a row
- A vertical set is called a column
- First subscript refers to the row number
- Second subscript refers to column number

Matrix notation

column 3

$$A_{(m,n)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} \dots & a_{mn} \end{bmatrix}$$

row 2

- It has the dimensions m by n (m x n)

$$A_{(3,4)} = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 0 & 2 & 1 \\ 5 & 1 & 1 & 7 \end{bmatrix}$$

$$a_{11} = 2, a_{12} = 1, a_{13} = 3, a_{14} = 0$$
$$a_{21} = 4, a_{22} = 0, a_{23} = 2, a_{24} = 1$$
$$a_{31} = 5, a_{32} = 1, a_{33} = 1, a_{34} = 7$$

Types of matrices

1/ Row vector m=1

$$R_{(1,n)} = [r_1 \quad r_2 \quad \ldots\ldots. \quad r_n]$$

$$R_{(1,4)} = [3 \quad -2 \quad 4 \quad 1]$$

2/ Column vector n=1

$$\underset{(m\times 1)}{C} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

$$C_{(3,1)} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

3/ Square Matrix m=n

$$A_{(n,n)} = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & a_{n3}\ldots & a_{nn} \end{bmatrix}$$

$$A_{(3,3)} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & 2 & 7 \end{bmatrix}$$

Some other Matrices

• Zero Matrix	المصفوفة الصفرية
• Transpose of a matrix	المصفوفة الممحورة
• Symmetric matrix	المصفوفة المتماثلة
• Diagonal matrix	المصفوفة القطرية
• Identity matrix	مصفوفة الوحدة
• Upper triangular matrix	مصفوفة مثلث علوي
• Lower triangular matrix	مصفوفة مثلث سفلي

Zero Matrix

Every element of a matrix is zero, it is called a zero matrix, i.e.,

$$O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ . & . & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

- The symbol O is used to denote the zero matrix.

Transpose of a matrix

The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T or A').

$$A_{(m,n)} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix} \Rightarrow A^T = A' = \begin{bmatrix} a_{11} & a_{21} & . & . & . & a_{m1} \\ a_{12} & a_{22} & . & . & . & a_{m2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{1n} & a_{2n} & . & . & . & a_{nm} \end{bmatrix}$$

e.g.

$$A_{(2,3)} = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 2 & 7 \end{bmatrix} \Rightarrow A^T_{(3,2)} = A'_{(3,2)} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \\ 3 & 7 \end{bmatrix}$$

Symmetric matrix

- A symmetric matrix is a square matrix which satisfy the following property:

$$a_{ij} = a_{ji} \ \ for\ all\ i's\ and\ \ j's.$$

- A symmetric matrix satisfy that : $A = A^T$

Example: the following matrix is symmetric

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & 9 \end{bmatrix} \qquad A^T = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & 9 \end{bmatrix} = A$$

$a_{12} = a_{21}$, $a_{13} = a_{31}$

$a_{23} = a_{32}$

Diagonal matrix

- A Diagonal matrix is a square matrix where all elements off the main diagonal are zero.

$$a_{ij} = a_{ji} = 0 \ \textit{for all } i\text{'s and } j\text{'s with } i \neq j$$

$$D_{(n,n)} = \begin{bmatrix} a_{11} & 0 & 0\ldots & 0 \\ 0 & a_{22} & 0\ldots & 0 \\ . & . & . & . \\ 0 & 0 & 0\ldots & a_{nn} \end{bmatrix}$$

Example:

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrix

- An identity matrix is a diagonal matrix where all elements of the main diagonal are one.

$$a_{ii} = 1 \, \forall i \;\; and \; a_{ij} = 0 \;\; for \, all \, i \neq j$$

$$I_n = \begin{bmatrix} 1 & 0 & 0\ldots & 0 \\ 0 & 1 & 0\ldots & 0 \\ . & . & . & . \\ 0 & 0 & 0\ldots & 1 \end{bmatrix}$$

Examples:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Upper triangular matrix

- An upper triangular matrix is a square matrix where all elements below the main diagonal are zero.

$$a_{ij} = 0 \ for\ all\ i > j$$

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Examples:

$$U_3 = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

$$U_4 = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Lower triangular matrix

- A Lower triangular matrix is a square matrix where all elements above the main diagonal are zero.

$$a_{ij} = 0 \; for \; all \; i < j$$

$$L = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}$$

Examples:

$$L_3 = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 7 & 3 \end{bmatrix} \qquad L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & 1 & 8 & 0 \\ 1 & 4 & 6 & 9 \end{bmatrix}$$

Usual operations on matrices

Addition and Subtraction of matrices

Matrices can be added or subtracted if they have the same order. Corresponding entries are added (or subtracted).

$$A_{mn} \pm B_{mn} = C_{mn}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} (a_{11} \pm b_{11}) & \cdots & (a_{1n} \pm b_{1n}) \\ (a_{21} \pm b_{21}) & \cdots & (a_{2n} \pm b_{2n}) \\ \vdots & & \vdots \\ (a_{m1} \pm b_{m1}) & \cdots & (a_{mn} \pm b_{mn}) \end{bmatrix}$$

Examples

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Find, if possible,

(1) A + B (2) A – C (3) B - A

(4) A + D (5) D – C (6) D - A

(7) C + D (8) C – D (9) D - B

Usual operations on matrices

Scalar Multiplication of matrices:

Multiplication of a matrix A by a scalar α is obtained by multiplying every element of A by α:

$\alpha A = [\alpha a_{ij}]$.

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & . & . & . & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & . & . & . & \alpha a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha a_{m1} & \alpha a_{m2} & . & . & . & \alpha a_{mn} \end{bmatrix}$$

Examples:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \qquad 3A = \begin{bmatrix} 3 & 6 \\ 3 & 9 \\ 3 & 15 \end{bmatrix} \qquad 5A = \begin{bmatrix} 5 & 10 \\ 5 & 15 \\ 5 & 25 \end{bmatrix}$$

Usual operations on matrices

Multiplication of two matrices:

Multiplication of a matrix *A* by a matrix B is possible if the number of columns of *A* is equal to the number of rows of B.

$$A_{m \times n} \; B_{n \times p} = C_{m \times p}$$

Example:

$$A_{(2,3)} \times B_{(3,4)} = C_{(2,4)}$$

Examples

Consider

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \qquad B=\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix} \qquad C=\begin{bmatrix}1 & 2\\ 3 & 0\\ 4 & 5\end{bmatrix}$$

1/ $$A\times B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\times\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}=\begin{bmatrix}1\times3+2\times2 & 1\times1+2\times5\\ 3\times3+4\times2 & 3\times1+4\times5\end{bmatrix}$$

2/ $$B\times A=\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}\times\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}\neq A\times B$$

3/ $$C\times B=\begin{bmatrix}1 & 2\\ 3 & 0\\ 4 & 5\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}=\begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

Example of use

- Let's show, by a real-life example, why matrix multiplication is defined in such a way.

- Ahmad, Muhammad and Nayef three students in IMAM University invest their money in the same stocks but with different quantities. Ahmad's, Muhammad's and Nayef's stock holding are given by the matrix:

$$M = \begin{array}{c} \\ Ahmad \\ Muhammad \\ Nayef \end{array} \begin{array}{ccc} Q_1 & Q_2 & Q_3 \\ 20 & 30 & 10 \\ 10 & 20 & 10 \\ 40 & 0 & 10 \end{array}$$

Example of use (continued)

- Now suppose that the prices of three types of stocks are respectively 25 SAR, 22 SAR and 30 SAR.

- Let the price matrix be $P = \begin{bmatrix} 25 \\ 22 \\ 30 \end{bmatrix}$

- Then the amount that the first student will pay is just the first entry of the product.

$$M \times P = \begin{bmatrix} 20 & 30 & 10 \\ 10 & 20 & 10 \\ 40 & 0 & 10 \end{bmatrix} \begin{bmatrix} 25 \\ 22 \\ 30 \end{bmatrix} = \begin{bmatrix} 1460 \\ 990 \\ 1300 \end{bmatrix}$$

Matrix Properties

- Matrix addition is commutative : $A + B = B + A$
- Matrix addition is associative: $A + (B + C) = (A + B) + C$
- Matrix addition is distributive : $\alpha(A + B) = \alpha A + \alpha B$

where α is a scalar.

- Matrix multiplication is associative : $(A.B).C = A.(B.C)$
- Matrix multiplication is distributive : $(A + B).C = A.C + B.C$
- Multiplication is generally not commutative : $A.B \neq B.A$

Time to Review !

✓ Matrices are used to store information. This information is written in a rectangular arrangement of rows and columns.

✓ Matrices can be added or subtracted if they have the same order.

✓Multiplication of a matrix A by a matrix B is possible if the number of columns of A is equal to the number of rows of B.

we will see in the next unit

- ✓ Determinant of matrices
- ✓ Some Properties of Determinants
- ✓ Some applications of determinant of matrices