Al-Imam Muhammad Ibn Saud Islamic University
College of Economics and Administration Sciences
Department of Finance and Investment

جامعة الإمام محمد بن سعود الإسلامية
كلية الاقتصاد والعلوم الإدارية
قسم التمويل والاستثمار

Course	**Financial Mathematics**
Unit course	**FIN 118**
Number Unit	**3**
Unit Subject	**Derivability** **Critical points** **Inflection point**

Dr. Lotfi Ben Jedidia
Dr. Imed Medhioub

We will see in this unit

1. Derivative of function
2. How to compute derivative?
3. Some rules of differentiation
4. Second and higher derivatives
5. Interpretation of the derivative
6. Critical points
7. Inflection point

LEARNING OUTCOMES

At the end of this chapter, you should be able to:

1. Understand what is meant by “Differentiation and Derivative of function”.
2. Compute these derivatives.
3. Apply some rules of differential calculus that are especially useful for decision making.
4. Find the critical & inflection points of a function.
5. Apply derivatives to real world situations in order to optimize unconstrained problems, especially economic and finance.

Derivative of Function

Definition1:

Differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x.

Definition2:

This rate of change is called the derivative of y (the function) with respect to x. The derivative gives the value of the slope of the tangent line to a curve at a point (rate of change).

Definition3:

The slope of the tangent line is very close to the slope of the line through (a, f(a)) and a nearby point on the graph, for example (a + h, f(a + h)).

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

How to compute derivative ?

To compute the derivative of a function, we can use four steps.

Step 1 : compute $f(x+h)$

Step2 : compute $f(x+h)-f(x)$

Step 3 : compute $\frac{f(x+h)-f(x)}{h}$

Step 4 : compute $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

How to compute derivative ?

Example1:

Find the derivative of $f(x) = 3x^2 + 5$

Step 1 : $f(x+h) = 3(x+h)^2 + 5 = 3x^2 + 6xh + 3h^2 + 5$

Step2 : $f(x+h) - f(x) = 3h^2 + 6xh$

Step 3 : $$\frac{f(x+h) - f(x)}{h} = \frac{3h^2 + 6xh}{h} = 3h + 6x$$

Step 4 : $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3h + 6x = 6x$$

$$f(x) = 3x^2 + 5 \Rightarrow f'(x) = 6x$$

Some rules of differentiation

To simplify the determination of derivatives we can use the following rules.

1/ $\frac{d}{dx}k = (k)' = 0$

2/ $\frac{d}{dx}cx = (cx)' = c$

3/ $\frac{d}{dx}x^n = \left(x^n\right)' = nx^{n-1}$

4/ $\frac{d}{dx}cx^n = \left(cx^n\right)' = ncx^{n-1}$

5/ $\frac{d}{dx}\sqrt{x} = \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$

6/ $\frac{d}{dx}\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$

7/ $\frac{d}{dx}\ln(x) = (\ln(x))' = \frac{1}{x}$

8/ $\frac{d}{dx}e^x = \left(e^x\right)' = e^x$

Some rules of differentiation

9/ $$\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$$

10/ $$\left(f(x) \times g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

11/ $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

12/ $$\left(f^n(x)\right)' = nf'(x)f^{n-1}(x)$$

13/ $$\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)}$$

14/ $$\left(e^{f(x)}\right)' = f'(x)e^{f(x)}$$

Derivative of Function

Examples:

Find the derivatives of the functions

1/ $f(x)=2x+1$

2/ $f(x)=3x^5-2x^3+7x-10$

3/ $f(x)=\left(x^2+7x-10\right)^3$

4/ $f(x)=\dfrac{x^2-1}{x^2+1}$

5/ $f(x)=\ln\left(x^2-1\right)$

6/ $f(x)=e^{\left(x^2-1\right)}$

Second and Higher Derivatives

Given a function f we defined first derivative of the function as:

$$f^{(1)}(x) = f^{'}(x) = \lim_{h \to 0} \frac{f\ (x+h) - f\ (x)}{h}$$

The second derivative is obtained by:

$$f^{(2)}(x) = f^{''}(x) = \lim_{h \to 0} \frac{f^{(1)}(x+h) - f^{(1)}(x)}{h}$$

And the n-th derivative is obtained by:

$$f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

Second and Higher Derivatives

Example:

1/ the function: $f(x) = 4x^5 - 3x^2 + 2x - 10$

2/ first derivative : $f^{(1)} = f'(x) = 20x^4 - 6x + 2$

3/ second derivative : $f^{(2)} = f''(x) = 80x^3 - 6$

4/ third derivative : $f^{(3)} = f'''(x) = 240x^2$

5/ fourth derivative : $f^{(4)} = f''''(x) = 480x$

6/ fifth derivative : $f^{(5)} = f'''''(x) = 480$

Interpretation of the derivative

First derivative:

- If $f'(x) > 0$ in a particular interval, then the function is increasing in that particular interval.
- If $f'(x) < 0$ in a particular interval, then the function is decreasing in that particular interval.

Second derivative:

- If $f''(x) > 0$ in a particular interval, then the graph of the function is concave upward or strictly convex "$\bigcup$" in that particular interval.
- If $f''(x) < 0$ in a particular interval, then the graph of the function is concave downward "$\bigcap$" or strictly concave in that particular interval.

Interpretation of the derivative

Example1:

Find where the function $f(x) = -x^3 + 1.5x^2 + 6x - 9$

is increasing and where it is decreasing.

<u>Solution:</u>

$$f'(x) = -3x^2 + 3x + 6 = -3(x+1)(x-2)$$

$$f'(x) = 0 \Rightarrow x = -1 \text{ or } x = 2$$

x		-1		2	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘		↗		↘

Thus the function is increasing on [-1;2] and it is decreasing on]-∞, -1[and]2, +∞[.

Interpretation of the derivative

Example2:

Find where the function $f(x) = -x^3 + 1.5x^2 + 6x - 9$ is strictly concave and where it is strictly convex.

Solution: $f''(x) = -6x + 3;\ f''(x) = 0 \Rightarrow x = 1/2$

x		1/2	
$f''(x)$	$+$	●	$-$
$f(x)$	$\cup$		$\cap$

Thus the function is convex on]-∞, 0.5[and concave on]0.5, +∞[.

Critical points: local extrema

Definition 1:

A critical point of a function of a single real variable $f(x)$ is a value x_0 in the domain of f where either the function is not differentiable or its derivative is 0, $f'(x)=0$.

- ✓ The point $M=(x_0, f(x_0))$ is a local minimum if

$$f'(x_0)=0 \; and \; f''(x_0)>0$$

- ✓ The point $M=(x_0, f(x_0))$ is a local maximum if

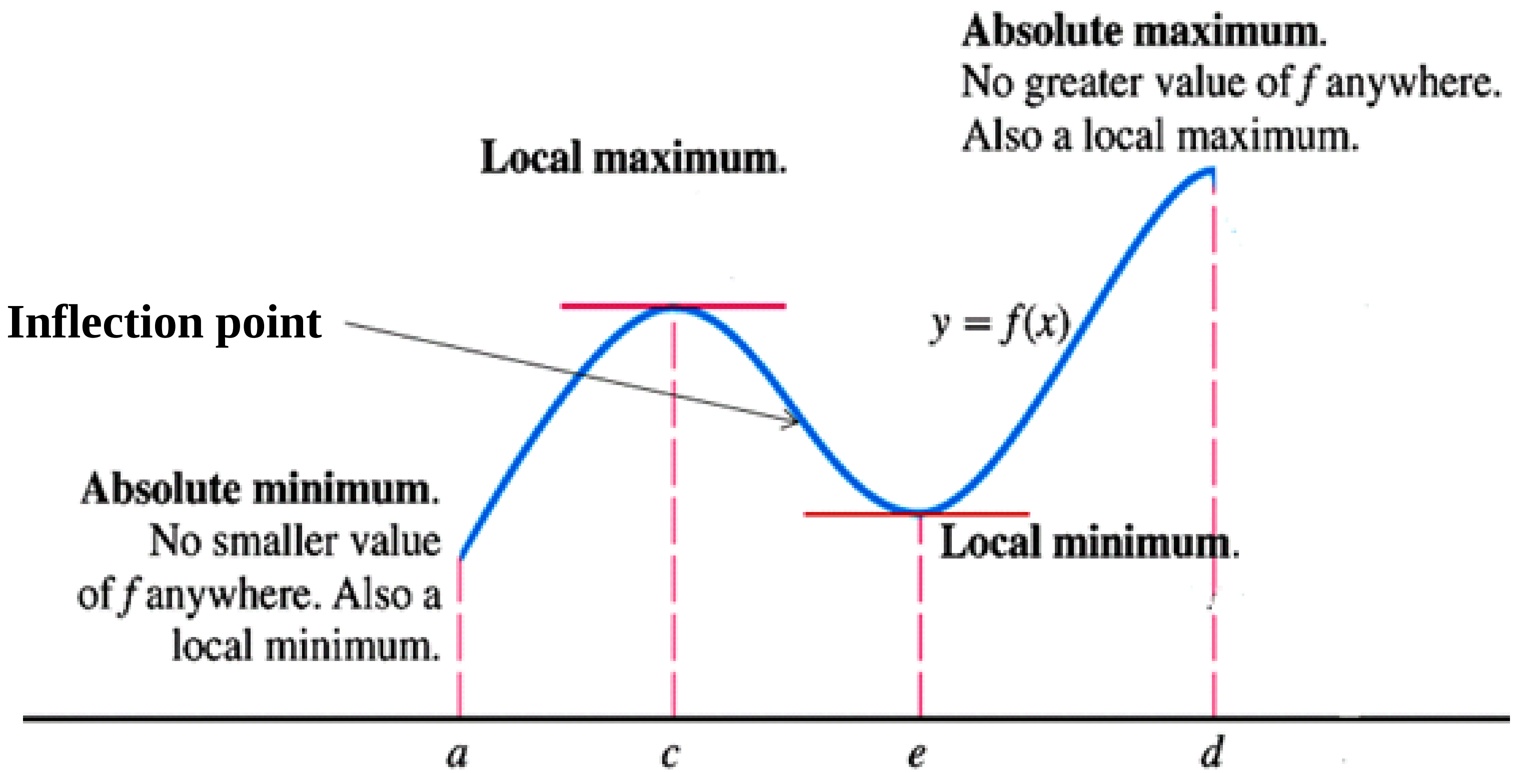
$$f'(x_0)=0 \; and \; f''(x_0)<0$$

Inflection point

Definition2 :

The point $M=(x_0, f(x_0))$ is an inflection point if the second derivative of the function changes signs from $+$ to $-$ or $-$ to $+$, i.e. the curve changes from concave upward to concave downward or from concave downward to concave upward at M.

Inflection point



Critical points

Example1: maximum

Find the critical point of the following quadratic function: $f(x) = -x^2 + 2x + 3$

Step 1 : $f'(x) = -2x + 2$ and $f''(x) = -2$

Step 2 : $f'(x) = 0 \Rightarrow -2x + 2 = 0, \textit{then} \quad x = 1$

Step 3 : we find the sign chart table.

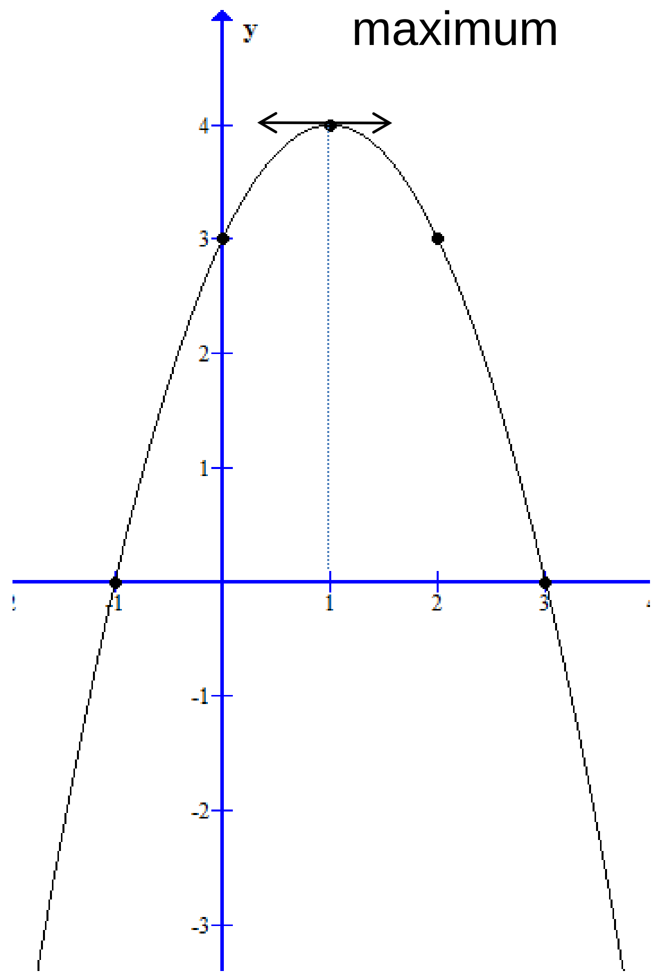
x	$-\infty$		1		$+\infty$
$f'(x)$		$+$	●	$-$	
$f(x)$	$-\infty$	↗	4	↘	$-\infty$
$f''(x)$		$-$		$-$	

Critical points

Graph of the function

- M correspond to the vertex point (see the chapter 2)
- X = -1 and x = 3 are the roots of the equation.
- The function is increasing in]-∞, 1[and decreasing in]1, +∞[

M(1,4) is an absolute maximum



Critical points

Example2: minimum

Find the critical point of the following quadratic function: $f(x) = x^2 + 2x + 3$

Step 1: $f'(x) = 2x + 2$ and $f''(x) = 2$

Step 2 : $f'(x) = 0 \Rightarrow 2x + 2 = 0, \textit{then} \ x = -1$

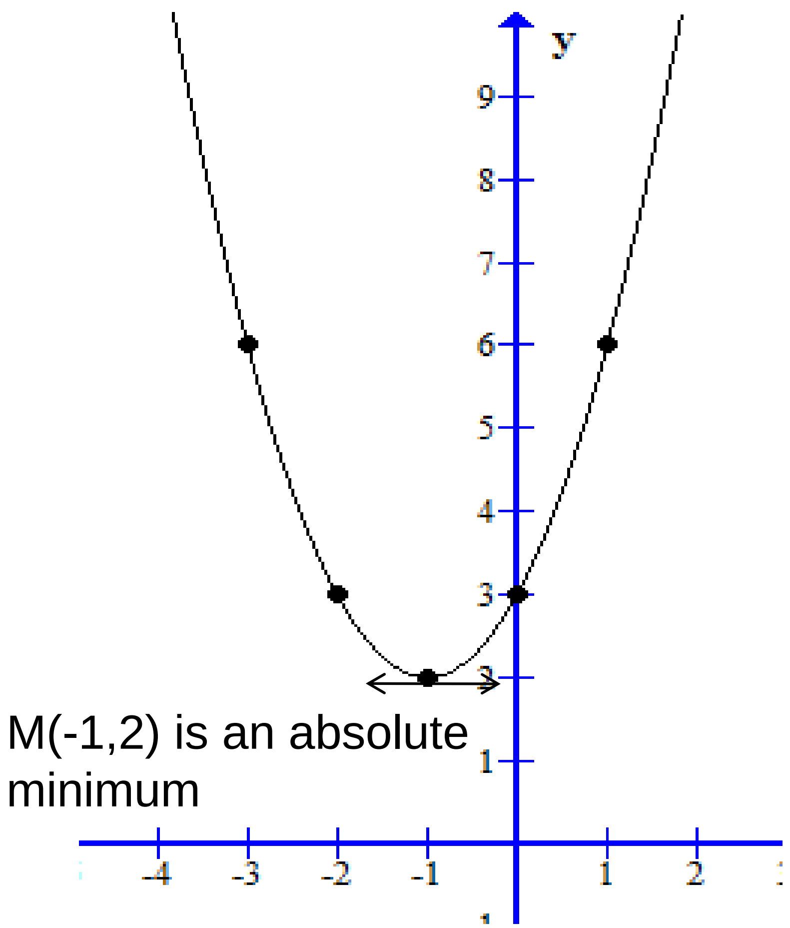
Step 3 : we find the sign chart table

x	$-\infty$		-1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	2	$\nearrow$	$+\infty$
$f''(x)$		$+$		$+$	

Critical points

Graph of the function

- M(-1,2) is an absolute minimum.
- It correspond to the vertex point (see chapter 2)
- There is no roots for the quadratic equation.
- The function is decreasing in]-∞, -1[and increasing in]-1, +∞[.



Critical & Inflection points

Example3: inflection point

Find the critical and inflection points of the following cubic function: $f(x) = x^3 - 3x + 1$

Step 1: $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

Step 2 : $f'(x) = 0 \Rightarrow 3(x^2 - 1) = 0, \, then \;\; x = \pm 1$

$$f''(x) = 0 \Rightarrow 6x = 0, \, then \;\; x = 0$$

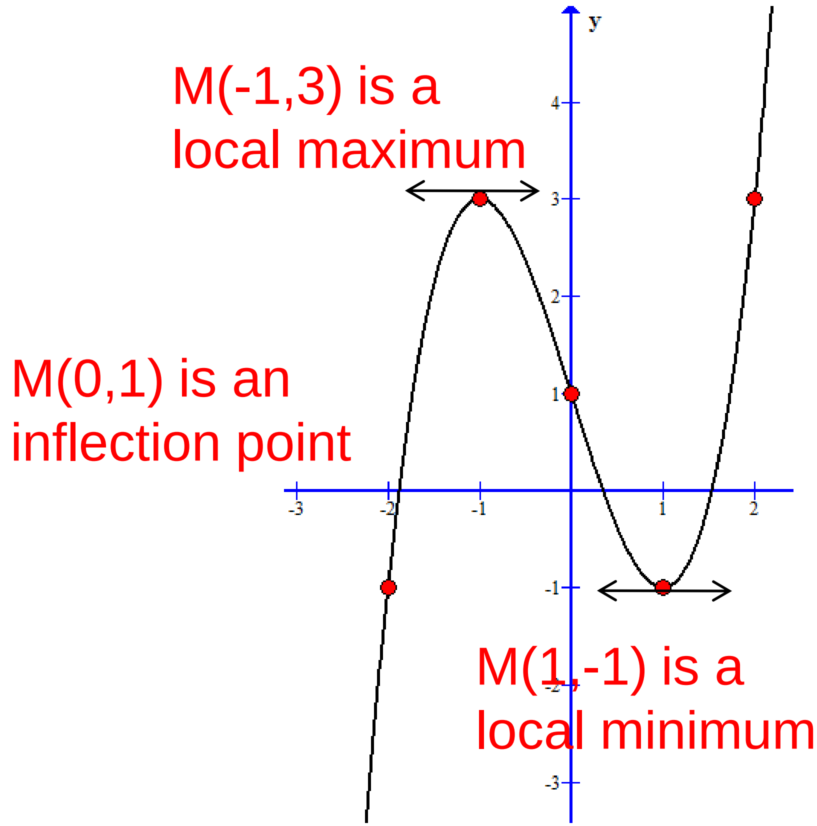
Step 3 : we find the sign chart table

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		$+$	0	$-$		$-$	0	$+$	
$f(x)$	$-\infty$	↗	3	↘	1	↘	-1	↗	$+\infty$
$f''(x)$		$-$		$-$	0	$+$		$+$	

Critical & Inflection points

Graph of the function

- The function is increasing on]-∞, -1[and]1, +∞[. But decreasing on]-1,1[
- It is concave on]-∞, 0[and convex on]0, +∞[.



Critical & Inflection points

Example 4:

Find the critical and the inflection points of the following function: $f(x) = x^3 + 2x^2 + x - 1$

Time to Review !

1. The derivative is the rate of change of a dependent variable with respect to an independent variable

2. The derivative of the function f at the point x = a is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Provided the limit exists.

3. First and Second derivatives of a function are useful tool to determine critical points and inflection point.

Time to Review !

f

- Increasing
- Decreasing
- Maximum or minimum value (when slope = 0)

f'

- Positive (above x axis)
- Negative (below x axis)
- Zero

f'' equals zero in x_0 and changes of signs,
M (x_0 , $f(x_0)$) is an inflection point

we will see in the next unit

1. The inverse process of differentiation: Integration.

2. The connection between integration and summation.

3. How to calculate area under a curve.

4. How to calculate area between two Curves.