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جامعة الإمام محمد بن سعود الإسلامية
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قسم التمويل والاستثمار

Course	**Financial Mathematics**
Unit course	**FIN 118**
Number Unit	**2**
Unit Subject	**Limits of Function** **Continuity of Function**

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We will see in this unit

1. Limit of function
2. Limits to the right, to the left
3. Computing limits
4. Properties of limits
5. Continuity of function
6. Essential discontinuities
7. Properties of continuous functions
8. Applications

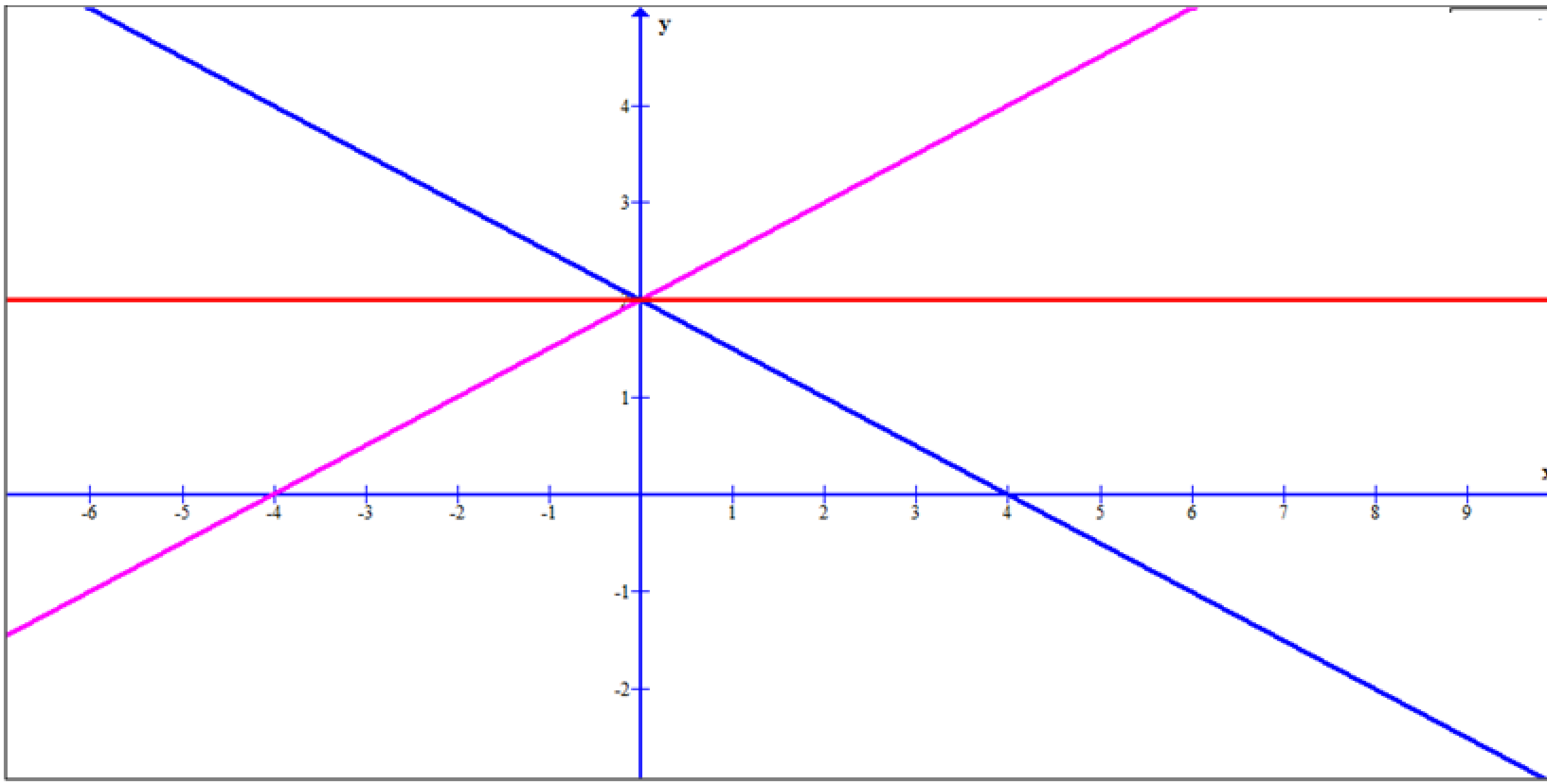
Learning Outcomes

At the end of this chapter, you should be able to:

1. Understand what is meant by “limit of function”.
2. Compute these limits.
3. Check existence of these limits.
4. Understand what is meant by a “continuous function”.
5. Check if a function is continuous graphically.
6. Check if a function is continuous algebraically.

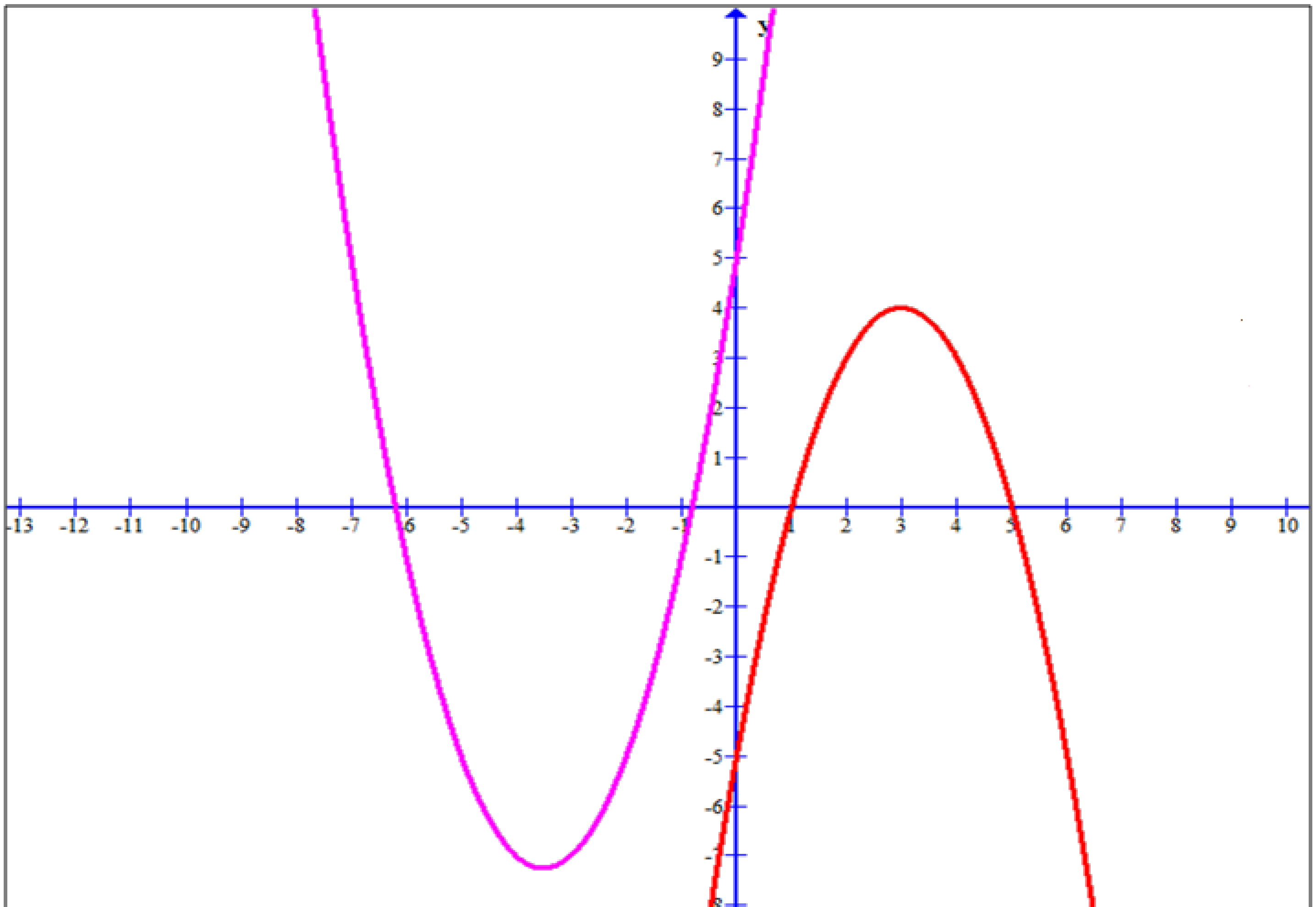
Limit of Function

Example 1: Deduce the value of each function when *x* gets close to +/- infinity



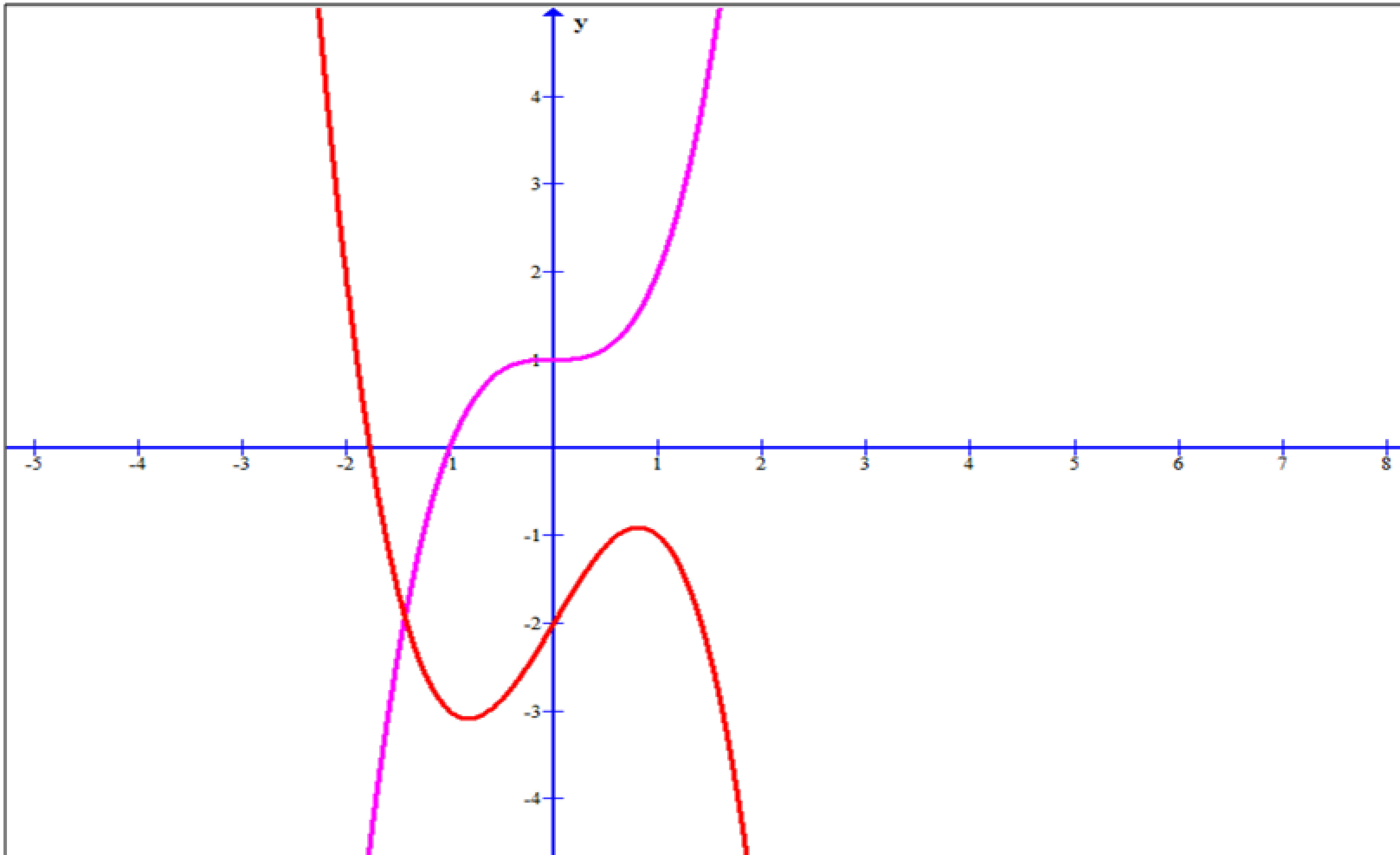
Limit of Function

Example 2: Deduce the value of each function when *x* gets close to +/- infinity



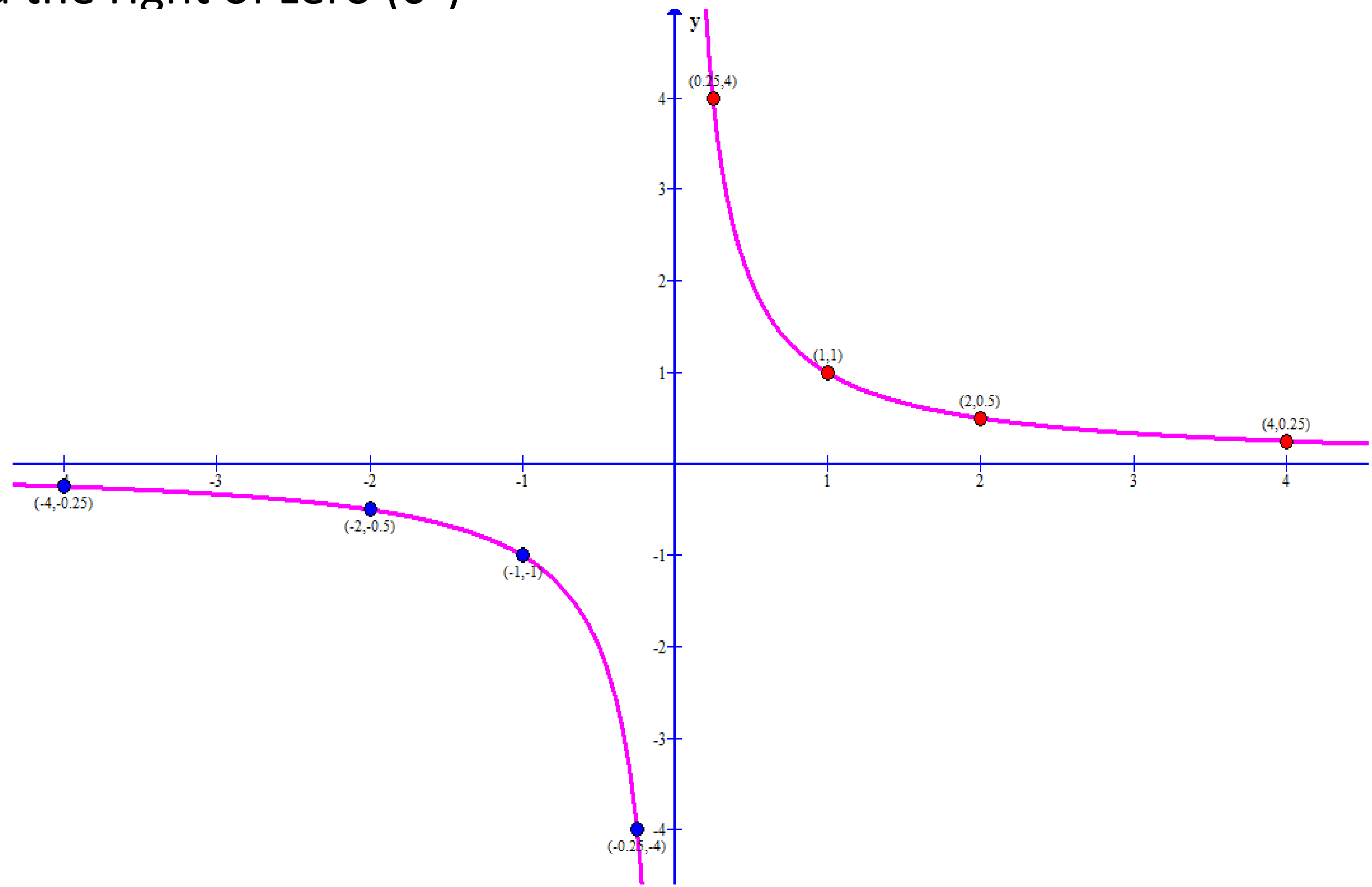
Limit of Function

Example 3: Deduce the value of each function when *x* gets close to +/- infinity



Limit of Function

Example 4: Deduce the value of the function when *x* gets close to +/- infinity and when x gets close to the left of zero (**0^-**) and the right of zero (0^+)



Limit of Function

Definition1 :

The limit of f(x), as x gets close to "a", equals L is written as: $\underset{x \to a}{Lim}\, f(x) = L$

Sometimes the values of function f tend to different limits as x approaches a number "a" from the left side and from the right side like the example 4.

Limits to the left, to the right

Left-Hand Limit (LHL) : $\boxed{\lim_{x\to a^-} f(x) = L_1}$

$$x \to a^- \Leftrightarrow x < a \Leftrightarrow x - a < 0$$

Right-Hand Limit (RHL) : $\boxed{\lim_{x\to a^+} f(x) = L_2}$

$$x \to a^+ \Leftrightarrow x > a \Leftrightarrow x - a > 0$$

Theorem

$\lim_{x \to a} f(x) = L$ if and only if:

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

* This theorem is used to verify whether a limit exists or not.

Computing limits

- To evaluate limit of a function, we substitute the value of (**a**) in the function.
- When the value of (**a**) does not in the domain of a function, then we must calculate the LHL and RHL.
- Sometimes we get an indeterminate form. So we must use one of three techniques: factoring or a property of limit or the conjugated form.

Computing limits by substitution

Example1: Find the limit of the function at x = 0, At x = 1, x = -2, and x = -3.

$$f(x) = 2x^2 + 3x - 1$$

Solution:

$$\lim_{x \to 0} 2x^2 + 3x - 1 = 2(0)^2 + 3(0) - 1 = -1$$

$$\lim_{x \to 1} 2x^2 + 3x - 1 = 2(1)^2 + 3(1) - 1 = 4$$

$$\lim_{x \to -2} 2x^2 + 3x - 1 = 2(-2)^2 + 3(-2) - 1 = 1$$

$$\lim_{x \to -3} 2x^2 + 3x - 1 = 2(-3)^2 + 3(-3) - 1 = 8$$

Computing limits by substitution

Example2:

let $$f(x)=\begin{cases} x & if\ \ x\le 0 \\ x+1 & if\ \ x>0 \end{cases}$$

Find the Left-hand Limit and the Right-hand Limit of the function at x=0.

What we can conclude ?

Solution

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x=0$$
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x+1=1$$

left-hand limit is different to right-hand limit then the limit at x = 0, **does not exist**.

Properties of limits

P1: If c is any real number, $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = M$

Then,

a) $\lim\limits_{x\to a}\left(f(x)+g(x)\right) = L+M$

b) $\lim\limits_{x\to a}\left(f(x)-g(x)\right) = L-M$

c) $\lim\limits_{x\to a}\left(f(x)\cdot g(x)\right) = L\cdot M$

d) $\lim\limits_{x\to a}\left(f(x)/g(x)\right) = L/M$, $(M \neq 0)$

e) $\lim\limits_{x\to a}\left(c\cdot f(x)\right) = c\cdot L$

f) $\lim\limits_{x\to a}\left(f(x)\right)^n = L^n$

g) $\lim\limits_{x\to a} c = c$

h) $\lim\limits_{x\to a} x = a$

i) $\lim\limits_{x\to a} x^n = a^n$

j) $\lim\limits_{x\to a} \sqrt{f(x)} = \sqrt{L}$, $(L > 0)$

Properties of limits

P2: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$

Then

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n}$$

P3: If

$$f(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}, \quad a_m \neq 0, b_n \neq 0$$

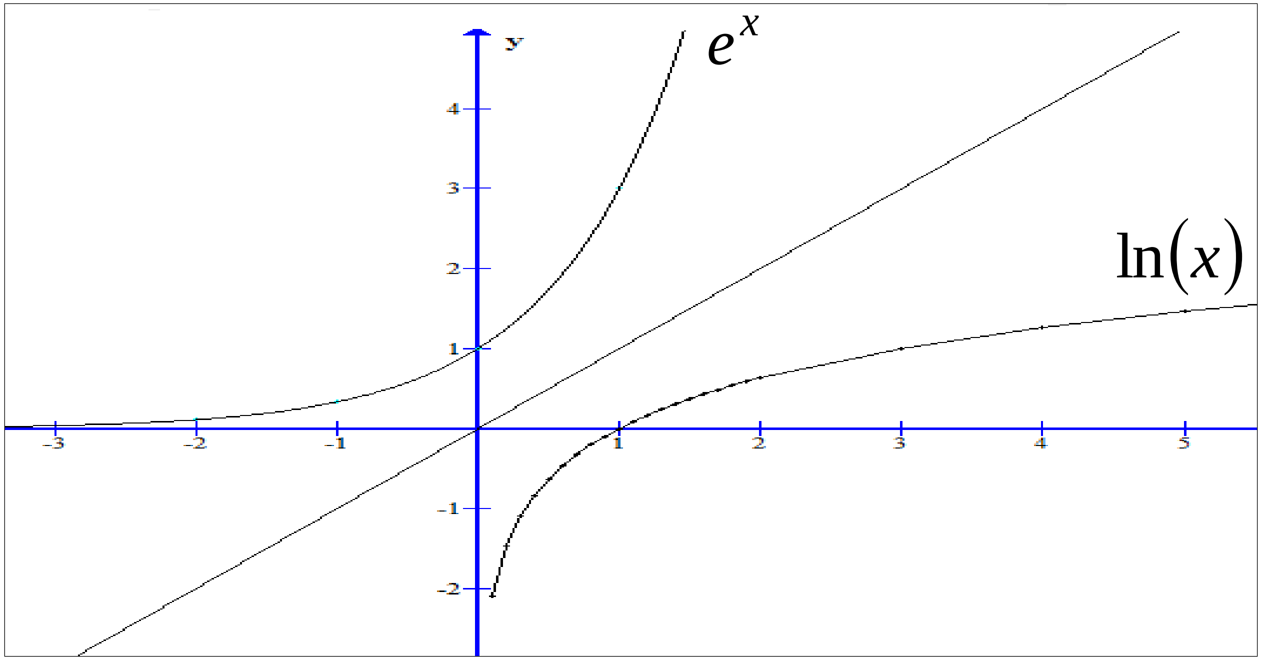
then

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{a_m x^m}{b_n x^n}}$$

Properties of limits

P4: If $f(x)=e^x$ then $\boxed{\lim_{x\to-\infty} f(x)=0}$ and $\boxed{\lim_{x\to+\infty} f(x)=+\infty}$

P5: If $f(x)=\ln(x)$ then $\boxed{\lim_{x\to0^+} f(x)=-\infty}$ and $\boxed{\lim_{x\to+\infty} f(x)=+\infty}$



Computing limits by using the highest degree term

Example2: Find the limit of the function when x gets close to (+/-) infinity.

$$f(x) = 2x^2 + 3x - 1$$

Solution:

$$\lim_{x \to -\infty} 2x^2 + 3x - 1 = \lim_{x \to -\infty} 2x^2 = 2(-\infty)^2 = +\infty$$

$$\lim_{x \to +\infty} 2x^2 + 3x - 1 = \lim_{x \to +\infty} 2x^2 = 2(+\infty)^2 = +\infty$$

Computing limits by using the highest degree term

Example3: Find the limit of the function when x gets close to (-)infinity.

$$f(x) = \frac{2x^2 + 3x - 1}{2x^3 + 2}$$

Solution:

$$\lim_{x \to -\infty} \frac{2x^2 + 3x - 1}{2x^3 + 2} = \lim_{x \to -\infty} \frac{2x^2}{2x^3} = \lim_{x \to -\infty} \frac{1}{x} = 0$$

Computing limits by using the highest degree term

Example4: Find the limit of the function when x gets close to (+)infinity.

$$f(x)=\frac{3x^2+3x-1}{4x^2+5x-4}$$

Solution:

$$\lim_{x\to+\infty}\frac{3x^2+3x-1}{4x^2+5x-4}=\lim_{x\to+\infty}\frac{3x^2}{4x^2}=\frac{3}{4}$$

Time to Review !

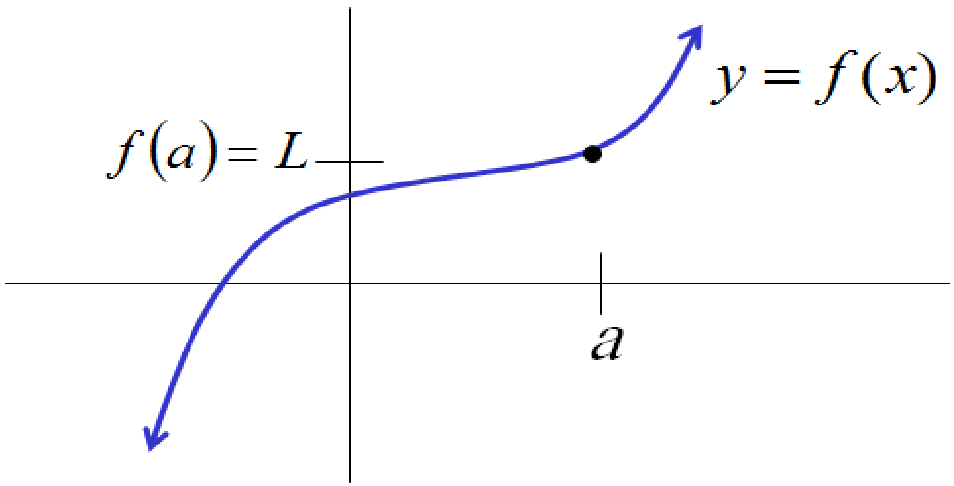
1. Remember well that the search for endings or limits is to know the behavior of the function when x gets close to a certain point or to infinity.
2. Before calculating limits, we search the domain of the function.
3. Computing limits can be achieved by substitution, by factoring, by using the highest degree term.
4. The limit exist if and only if LHL = RHL.

Continuity of Function

Definition1:

A function f is continuous if you can draw it in one motion without picking up your pencil.

Example:



Definition2:

A function f is continuous at a point if the limit is the same as the value of the function.

Continuity of Function

Definition3:

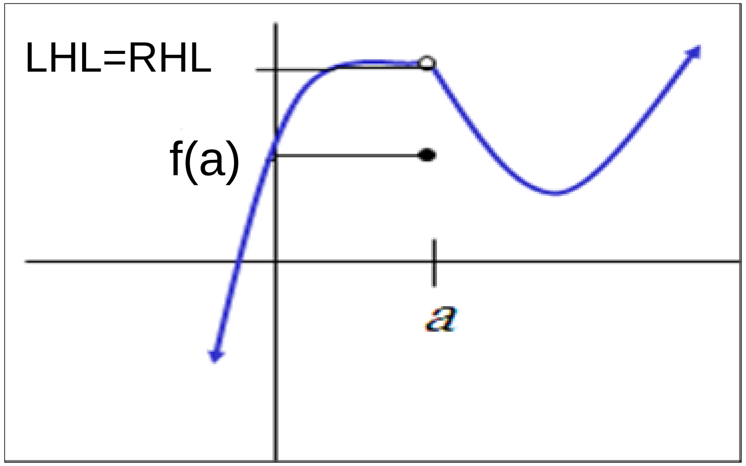
A function f is continuous at the point x=a if the following are true:

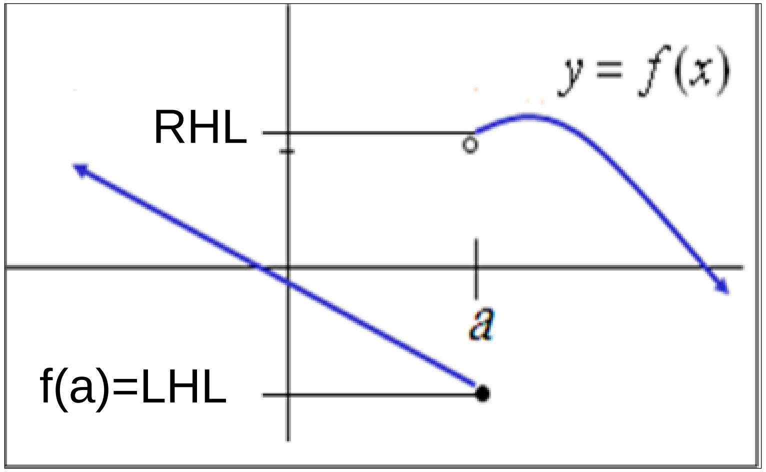
1. $f(a)$ *is defined*
2. $\lim\limits_{x\to a} f(x)$ *exist*
3. $\lim\limits_{x\to a} f(x) = f(a)$

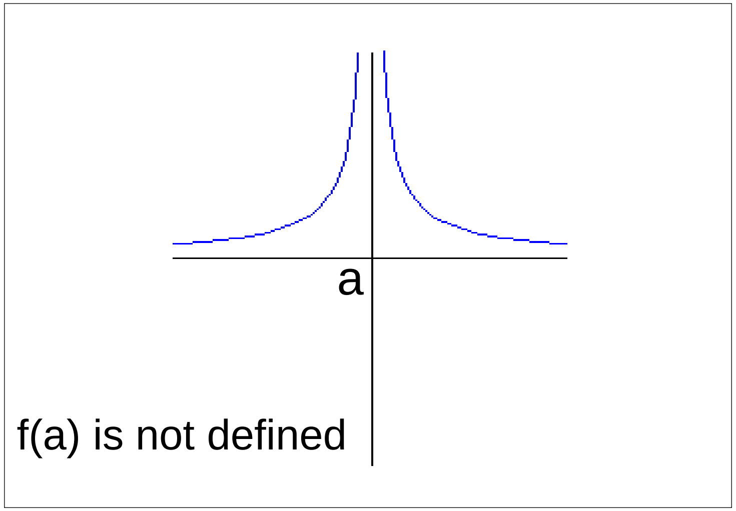
EX. The function $f(x)=3x^2+2x-2$ is continuous at x=0.

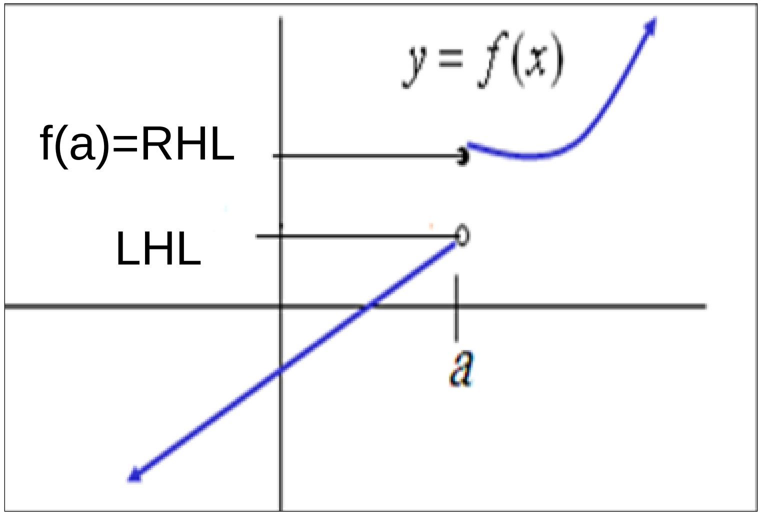
1. $f(0)=-2$ ☺
2. $\lim\limits_{x\to 0} f(x)=3(0)^2+2(0)-2=-2$ ☺
3. $\lim\limits_{x\to 0} f(x)=f(0)=-2$ ☺

Essential Discontinuities



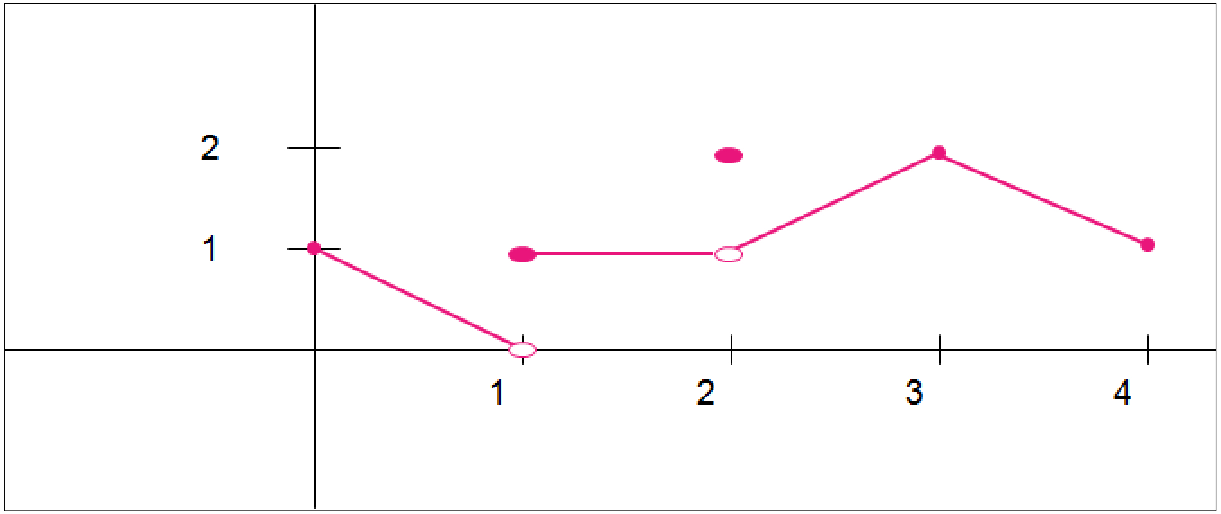






Continuity of Function

Example 2:



- This function has discontinuities at x=1 and x=2.
- It is continuous at x=3, because the two-sided limits match the value of the function.
- It is continuous at right of x=0 and left of x=4, because the one-sided limits match the value of the function.

Continuity of Function

Example 3:

Find discontinuities of the function

$$f(x)=\begin{cases} x & if\ \ x<0 \\ x^2 & if\ \ 0\leq x<2 \\ 8-x & if\ \ x\geq 2 \end{cases}$$

Solution:

the domain of the function is the real line. The discontinuities of the function may be at x=0 and/or x=2. why ?

Because the graph of f may have a break at x = 0 and or x = 2.

we will check the discontinuity, point by point.

Continuity of Function

Example 3: solution

$$f(x)=\begin{cases} x & \text{if } x<0 \\ x^2 & \text{if } 0\le x<2 \\ 8-x & \text{if } x\ge 2 \end{cases}$$

1/ at x = 0

* $f(0)=(0)^2=0$ ☺

* $\begin{cases} \lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} x=0 \\ \lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x^2=0 \end{cases}$ LHL = RHL=0 then $\lim\limits_{x\to 0} f(x)=0$ ☺

* $\lim\limits_{x\to 0} f(x)=f(0)=0$ ☺

The function f is continuous at x = 0.

Continuity of Function

Example 3: solution

$$f(x)=\begin{cases} x & if \ \ x<0 \\ x^2 & if \ \ 0\le x<2 \\ 8-x & if \ \ x\ge 2 \end{cases}$$

2/ at x = 2

* $f(2)=8-2=6$ ☺

* $\begin{cases} \lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-} x^2=(2)^2=4 \\ \lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+} 8-x=8-2=6 \end{cases}$

LHL $\neq$ RHL then $\lim\limits_{x\to 2} f(x)$ **does not exist**

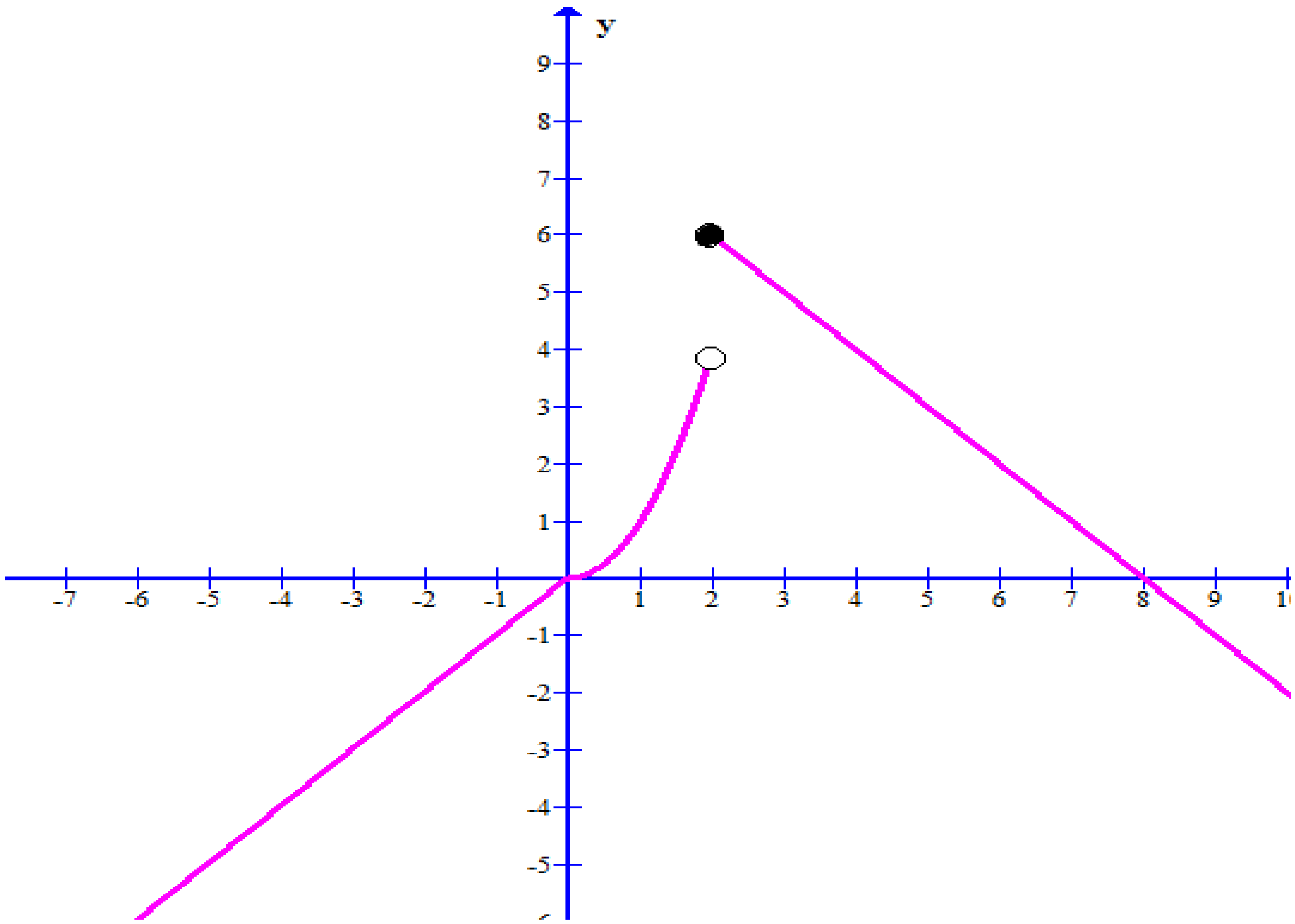
The function f is not continuous at x = 2.

!!! We say that the function f is continuous to the right point x= 2.

Continuity of Function

Example 3: continued

We can draw the graph of the function



Properties of continuous functions:

P1: If f and g are continuous functions at x = a then: $f \pm g$, $f \times g$, and $\dfrac{f}{g}$ (with $g(a) \neq 0$) are continuous functions at x = a.

P2: A polynomial function is continuous at every point. $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$

P3: A rational function is continuous at every point in its domain.

$$f(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}, \quad a_m \neq 0, b_n \neq 0$$

Continuity of Function

Examples:

1/ $f(x)=3x^4+2x^3-5x+10$ is a polynomial function, then it is continuous at every point in R.

2/ $f(x)=\dfrac{2x^3-5x+10}{x-2}$ is a rational function, then it is continuous at every point in R\{2}

3/ $f(x)=\sqrt{2x+1}$ is a continuous function in its domain $D_f = \left]\dfrac{-1}{2};+\infty\right[$

Time to Review !

1. A function is continuous if you can draw it in one motion without picking up your pencil.

2. A function is continuous at a point if the limit is the same as the value of the function.

3. All constant functions are continuous.

4. The following types of functions are continuous at every member in their domain: polynomial, rational, power, root, trigonometric, exponential, and logarithmic.

We will see in the next unit

1. Derivative of function
2. How to compute derivative?
3. Some rules of differentiation
4. Second and higher derivatives
5. Interpretation of the derivative
6. Critical points
7. Inflection point