Al-Imam Muhammad Ibn Saud Islamic University
College of Economics and Administration Sciences
Department of Finance and Investment

جامعة الإمام محمد بن سعود الإسلامية
كلية الاقتصاد والعلوم الإدارية
قسم التمويل والاستثمار

Course	**Financial Mathematics**
Unit course	**FIN 118**
Number Unit	**10**
Unit Subject	**Ordinary Annuity** **Annuity due**

Dr. Lotfi Ben Jedidia
Dr. Imed Medhioub

we will see in this unit

- ✓ Meant of simple Annuity
- ✓ Ordinary Annuity
- ✓ Annuity Due
- ✓ How to Calculate present and future values of each type of annuity.

LEARNING OUTCOMES

At the end of this chapter, you should be able to:

1. Distinguish between an ordinary annuity and an annuity due.

2. Calculate present and future values of each type of annuity.

3. Apply knowledge of annuities to solve a range of problems, including problems involving principal-and-interest loan contracts.

Introduction

In unit 9, we have seen that a single sum of money invested today for several periods will produce a higher future sum due to compounding effect. In this unit we attempt to see that the same phenomenon will occur for multiple stream of cash flow.

A multiple stream of cash flow that is made in an equal size and at a regular interval is known as simple annuity.

It exists four different forms for Simple annuity: Ordinary Annuity, Annuity Due (unit10), Deferred Annuity, and Perpetuity.

Ordinary Annuity

Definition: Ordinary Annuity is a series of equal cash payments or deposits made at the end of each compounding period.

Examples :

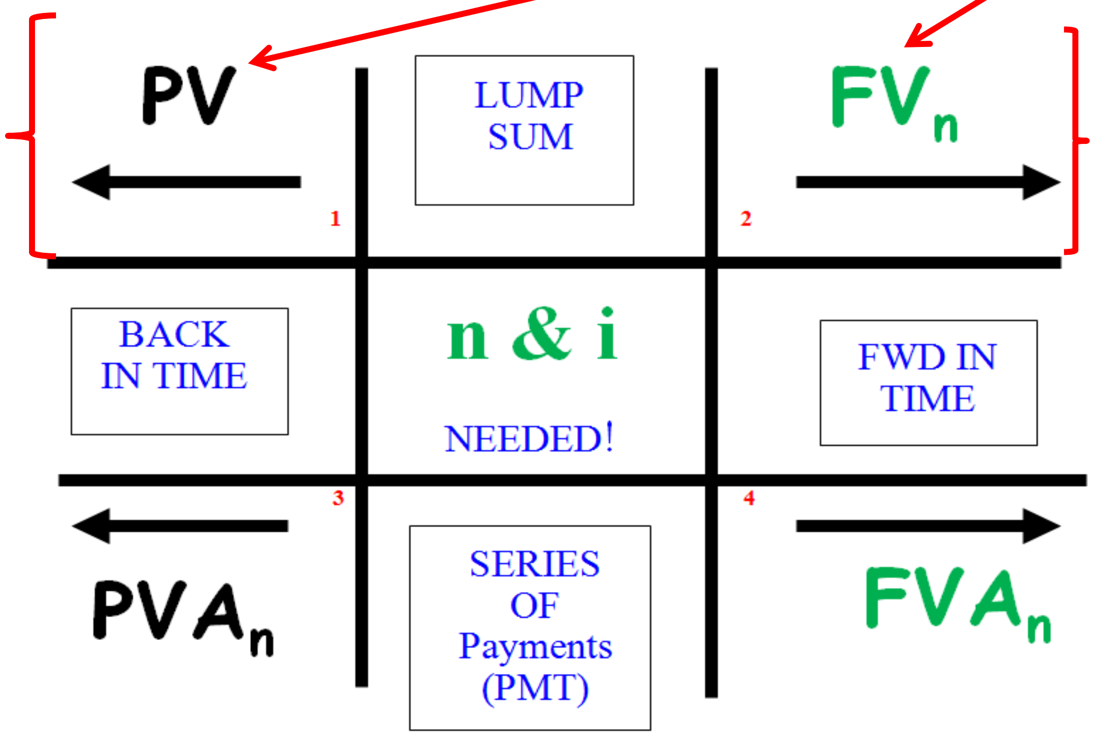
i/ When a particular individual buy a bond, he will receive equal semi-annual coupon interest payments over the life of the bond.

ii/ When a particular individual borrow money to buy for example a house or a car, he will pay a stream of equal payments at the end of each period of coverage.

Ordinary Annuity

Question: what is value of the sum of all payments now and at the end of period?

See unit 9



Future Value Annuity FVA_n

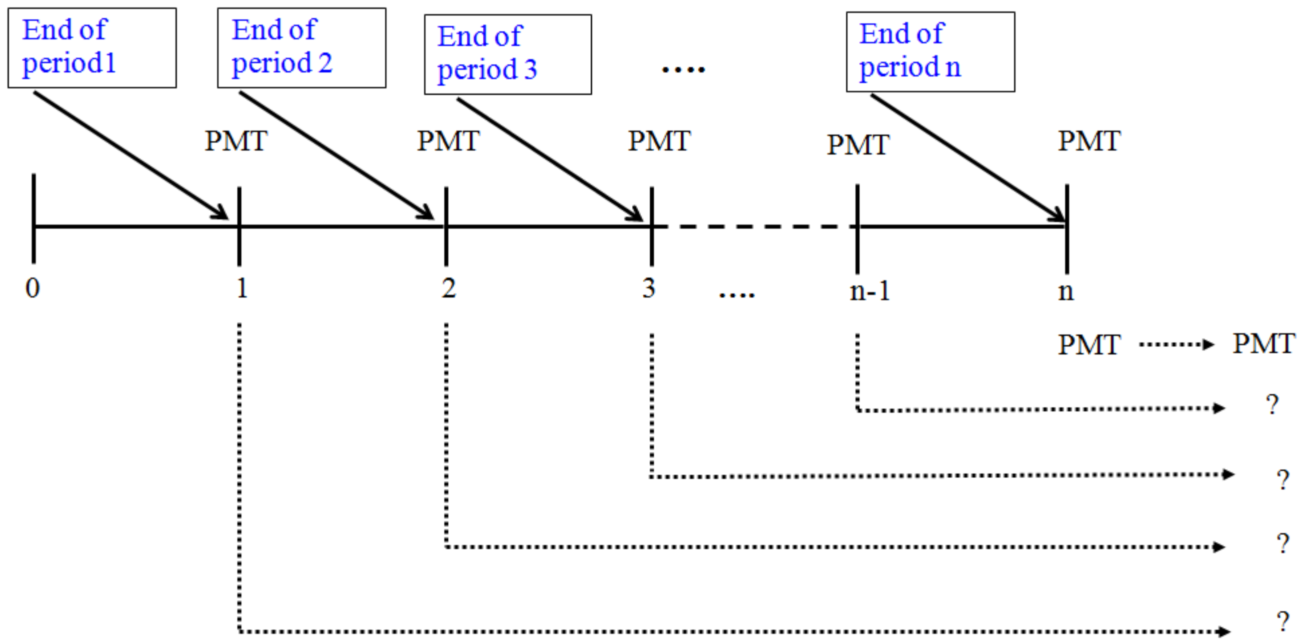
The future value annuity of an ordinary annuity is the sum of all regular equal payments and the compounded interest accumulated at the end of last period. It is determined as follow:

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- PMT: annuity payment deposited or received at the end of each period,
- i : interest rate per period,
- n : number of payments.

Future Value Annuity FVA_n

Proof :



$$FVA_n = PMT + PMT(1+i) + PMT(1+i)^2 + \cdots + PMT(1+i)^{n-1}$$

$$= PMT \times \left[1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}\right]$$

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

Future Value Annuity

More Examples

Example 1: Future Value Annuity

Suppose you plan to deposit $1000 annually into an account at the end of each of the next 7 years. If the account pays 12% annually, what is the value of the account at the end of 7 years?

Solution :

The general equation for a FV of an annuity is:

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

We have n = 7, i = 0.12 and PMT = 1000 then

$$FVA_7 = 1000\left[\frac{(1+0.12)^7 - 1}{0.12}\right] = 10089.01$$

* At the end of seventh year we will have $10089.01

Future Value Annuity

Example 2

- What is the future value of $5000 invested at the end of each year for 10 years if money earns 6% per annum?

$$FVA = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$$= 5000\left[\frac{(1+0.06)^{10} - 1}{0.06}\right] = 5000\left[\frac{0.7908}{0.06}\right] = 5000 \times (13.1808)$$

$$= \$65903.97$$

Present Value Annuity PVA_n

The present value annuity of an ordinary annuity is the sum of all regular equal payments discounted at a certain interest rate in at the end of each period. It is determined as follow:

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

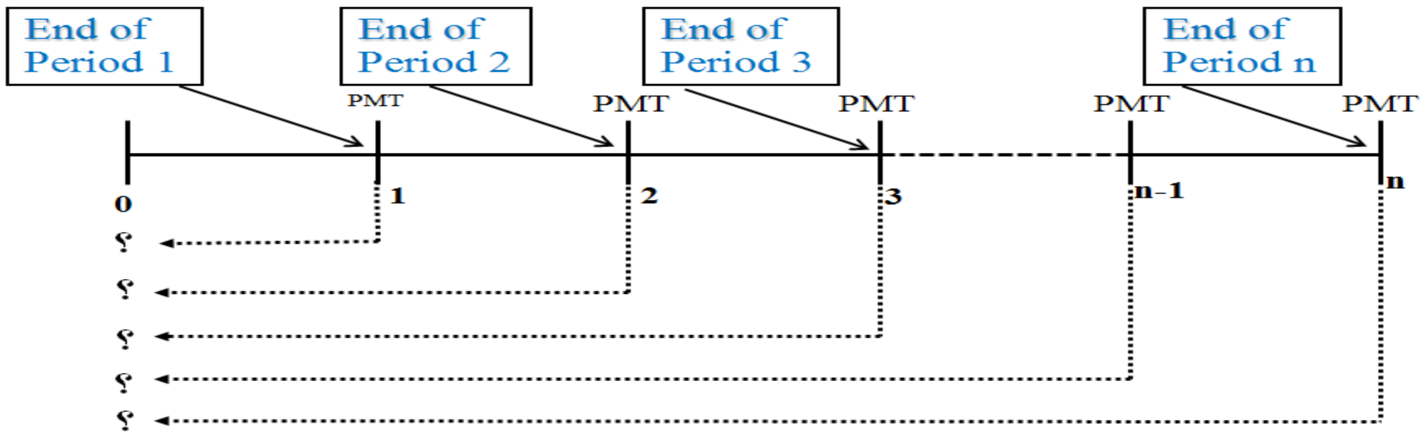
PMT: annuity payment deposited or received at the end of each period,

i : interest rate per period,

n : number of payments.

Present Value Annuity PVA_n

Proof :



$$PVA_n = \frac{PMT}{1+i} + \frac{PMT}{(1+i)^2} + \cdots + \frac{PMT}{(1+i)^n}$$

$$= \frac{PMT}{1+i} \times \left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} \right] =$$

$$= PMT(1+i)^{-1} \times \left[1 + (1+i)^{-1} + (1+i)^{-2} + \ldots + (1+i)^{-(n-1)} \right]$$

$$= PMT(1+i)^{-1} \left[\frac{1-(1+i)^{-n}}{1-(1+i)^{-1}} \right] = PMT \left[\frac{1-(1+i)^{-n}}{i} \right]$$

Present Value Annuity

More Examples

Example 1: Present Value Annuity

You plan to withdraw $1000 annually from an account at the end of each of the next 7 years. If the account pays 12% annually, what must you deposit in the account today?

Solution : The general equation for a PV of an annuity is:

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

We have n = 7, i = 0.12 and PMT = 1000 then

$$PVA_7 = 1000\left[\frac{1-(1+0.12)^{-7}}{0.12}\right] = 4563.76$$

At the beginning of the period we must have $4563.76

Present Value Annuity

More Examples

Example 2:

What is the present value of $5000 that will be invested at the end of each year for 10 years if money earns 6% per annum?

$$PVA = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 5000\left[\frac{1-(1+0.06)^{-10}}{0.06}\right] = 5000\left[\frac{0.4416}{0.06}\right] = 5000 \times (7.360087)$$

$$= \$36800.44$$

Annuity Due

Definition: Annuity Due is a series of equal cash payments or deposits made at <u>the beginning of each</u> compounding period.

Examples :

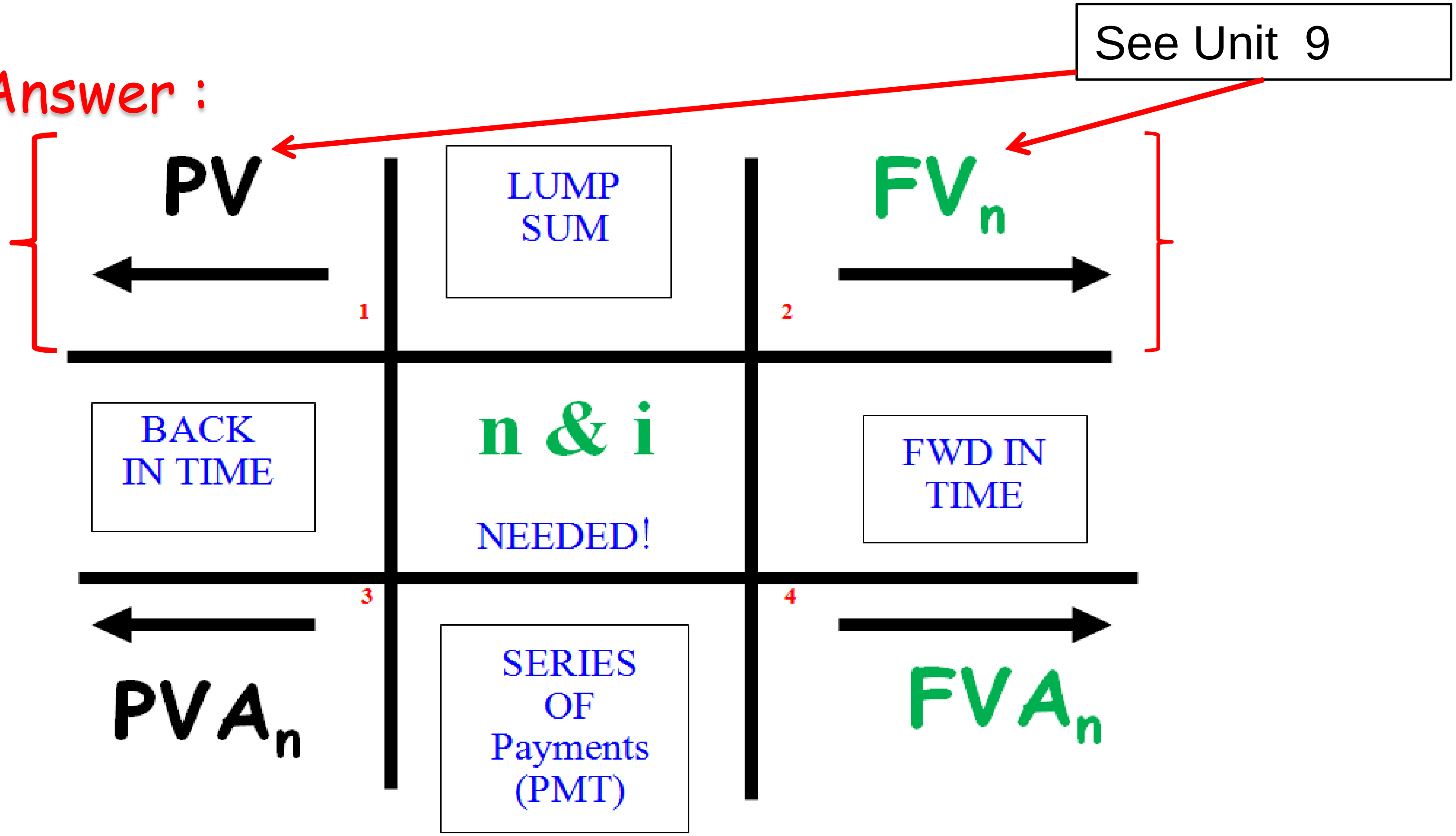
i/ When a particular individual make an apartment lease contract over a period of several years, he must paid at the beginning of each year an annual rent .

ii/ When a particular individual buy a car he must paid at the beginning of each year an annual insurance premium.

Annuity Due

Question: what is value of the sum of all payments now and at the end of period?

Answer :



Future Value Annuity FVA_n

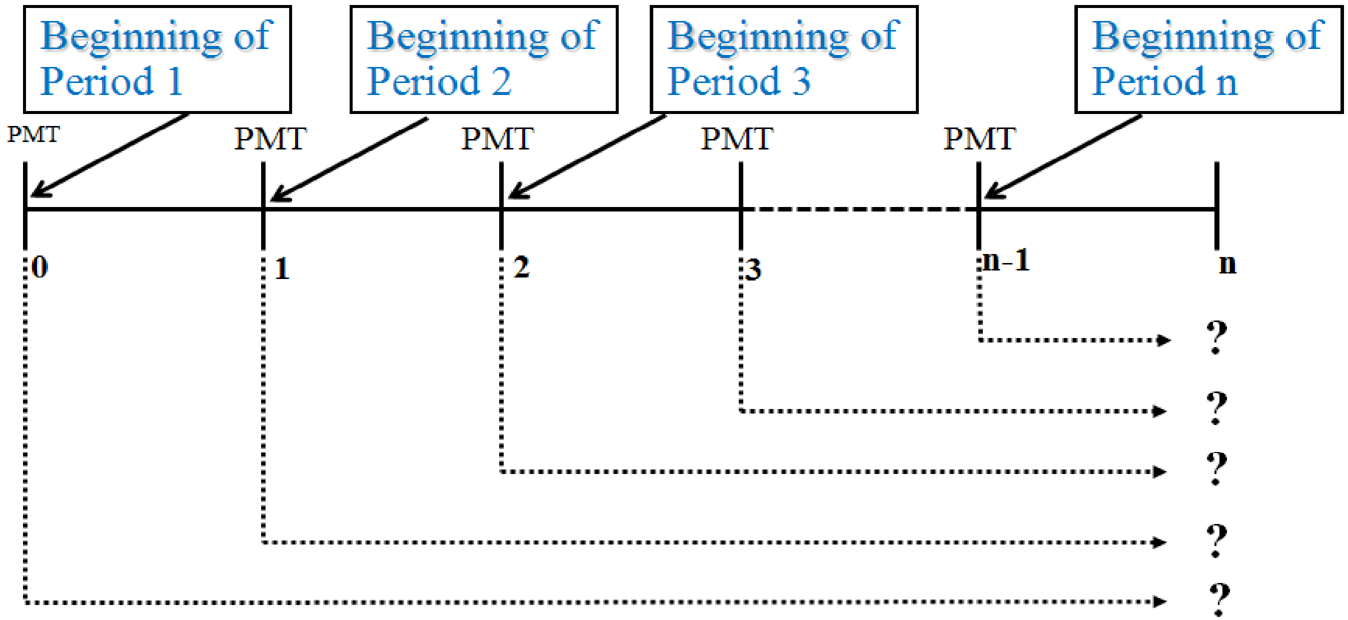
The future value annuity of an annuity due is the sum of all regular equal payments at the beginning of each period and the compounded interest accumulated at the end of last period. It is determined as follow:

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

- PMT: annuity payment deposited or received at the beginning of each period,
- i : interest rate per period,
- n : number of payments.

Future Value Annuity FVA_n

Proof :



$$FVA_n = PMT(1+i) + PMT(1+i)^2 + PMT(1+i)^3 \cdots + PMT(1+i)^n$$

$$= PMT(1+i) \times \left[1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}\right]$$

$$FVA_n = PMT(1+i)\left[\frac{(1+i)^n - 1}{i}\right] = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

Future Value Annuity

More Examples

Example 1: Future Value Annuity in Annuity Due

Suppose you plan to deposit $1000 annually into an account at the beginning of each of the next 7 years. If the account pays 12% annually, what is the value of the account at the end of 7 years?

Solution :

The general equation for a FV of an annuity is:

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

We have n = 7, i = 0.12 and PMT = 1000 then

$$FVA_7 = 1000\left[\frac{(1+0.12)^7 - 1}{0.12}\right](1+0.12) = 11299.69$$

At the end of seventh year we will have $11299.69

Future Value Annuity

More Examples

Example 2:

What is the future value of $5000 invested at the beginning of each year for 10 years if money earns 6% per annum?

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

$$FVA_{10} = 5000\left[\frac{(1+0.06)^{10} - 1}{0.06}\right](1+0.06)$$

$$= 5000\left[\frac{0.7908}{0.06}\right](1+0.06)$$

$$= \$69858.21$$

Present Value Annuity PVA_n

The present value annuity of an annuity due is the sum of all regular equal payments discounted at a certain interest rate in at the beginning of each period. It is determined as follow:

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

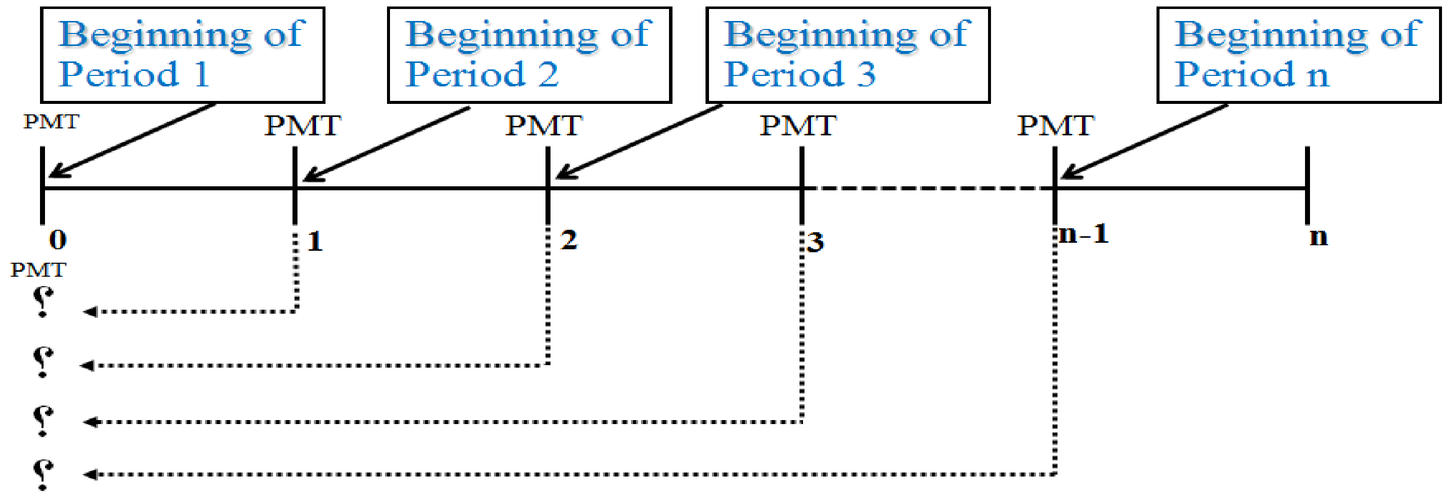
* PMT: annuity payment deposited or received at the beginning of each period,

* i : interest rate per period,

* n : number of payments.

Present Value Annuity PVA_n

Proof :



$$PVA_n = PMT + \frac{PMT}{1+i} + \frac{PMT}{(1+i)^2} + \cdots + \frac{PMT}{(1+i)^{n-1}}$$

$$= PMT \times \left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}}\right] =$$

$$= PMT \times \left[1 + (1+i)^{-1} + (1+i)^{-2} + \ldots + (1+i)^{-(n-1)}\right]$$

$$= PMT\left[\frac{1-(1+i)^{-n}}{1-(1+i)^{-1}}\right] = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

Present Value Annuity

More Examples

Example 1: Present Value Annuity

You plan to withdraw $1000 annually from an account at the beginning of each of the next 7 years. If the account pays 12% annually, what must you deposit in the account today?

Solution : The general equation for a PV of an annuity is:

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

We have n = 7, i = 0.12 and PMT = 1000 then

$$PVA_7 = 1000\left[\frac{1-(1+0.12)^{-7}}{0.12}\right](1+0.12) = 5111.41$$

At the beginning of the period we must have $5111.41

Present Value Annuity

More Examples

Example 2:

What is the present value of $5000 that will be invested at the beginning of each year for 10 years if money earns 6% per annum?

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

$$PVA_{10} = 5000\left[\frac{1-(1+0.06)^{-10}}{0.06}\right](1+0.06)$$

$$= 5000 \times (7.360087)(1.06)$$

$$= \$39008.46$$

Formulas

Time to Review!

Simple Annuity

Ordinary Annuity

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Annuity Due

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

Deferred Annuity

Perpetuity

That's All for 50% of simple Annuity !

we will see in the next unit

- ✓ Long term ordinary annuity
- ✓ Long term annuity due
- ✓ Amortization & sinking Funds
- ✓ Some real life examples